Some Results on semilinear spaces over special semiring

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Abstract

The main motive of this paper is to develop results for semimodules over special semiring called strict semi domain identical to linear spaces over fields. This paper is divided into two parts, in first part we have given some preliminaries. Also we obtain analogue of ring and module theoretic concepts for strict semirings. We prove that basis of a subspace of a semilinear space can be extended to the basis of semilinear space. In second part we prove rank-nullity theorem in semilinear spaces over special semiring called strict semidomain. To give examples of finite strict semirings, we have used Lattice theory.

Keywords: strict semiring, range, kernel, lattice, basis.

Introduction

Cuninghame evolved a theory over strict semirings similar to how linear algebra works over fields. [1]. A great deal of research on strict semirings is released in [2, 3, 4, 5, 6, 7, 8, 9]. Qian Shu and Wang proved some results about the dimensions of semilinear subspaces and direct sums in 2013[10]. Here we proved Rank-Nullity theorem for semimodules over special semiring called strict semidomain similar to that of linear spaces over fields.

Some preliminaries are given in part 2 and Rank-Nullity theorem in semilinear spaces over strict semiring is proved in part 3.

Preliminaries

Definition 2.1[11]: An non-empty set S with the binary operations "+" and "•" represented by (S, +, •) is called a semiring subject to the following conditions

i) Commutative monoid (S, +, 0) with identity member 0.

ii) A monoid with identity element 1 is (S, •, 1).

iii) (a + b) • c = a • c + b • c and a • (b + c) = (a • b) + (a • c), for all a, b, c ∈ $\mathbb{S}$.

Definition 2.2 [11, 12]: If for every a, b ∈ S, a • b = b • a, then the semiring (S, +, •) is commutative.

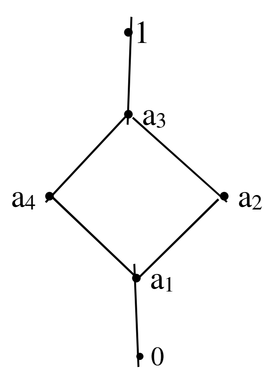
Note that unless otherwise mentioned $\mathbb{S}$ is a commutative semiring throughout this paper.

Example 2.1 [12]: Z_0^+, the set containing all positive integers with zero, is a commutative semiring.

Example 2.2 [12]: Q_0^+ , the set containing all positive rational numbers with zero is a commutative semiring.

Theorem 2.1 [12]: Let ($\mathcal{L}$, ∧, ∨) be a distributive lattice. Then $\mathcal{L}$ is a commutative semiring.

Example 2.3 [12]: Let $\mathbb{S}$ be a finite semiring, as indicated by the distributive Lattice below.

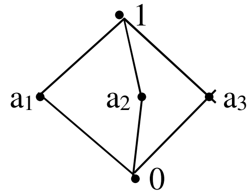


Corollary 2.1 [12]: Every chain lattice is a semiring.

Note 2.1 [12]: Every lattice is not a semiring.

Example 2.4 [12]: Following Lattice $\mathcal{L}$ doesnot hold distributive property. Hence $\mathcal{L}$ is not a semiring.

Hasse diagram:

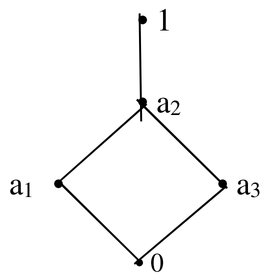


Definition 2.3 [11]: $\mathcal{T}$, subset of a semiring $\mathbb{S}$. If $\mathcal{T}$ is a semiring, then $\mathcal{T}$ is a subsemiring of $\mathbb{S}$.

Example 2.5 [12]: A subsemiring of the set of non-negative rational numbers is the set of non-negative integers.

Definition 2.4 [12]: Let $\mathbb{S}$ be a semiring then an element $a \in \mathbb{S}$, $a \neq 0$ is called zero divisor of $\mathbb{S}$ if $a \bullet b = 0$ for some $b \in \mathbb{S}$, $b \neq 0$.

Example 2.6: See the semiring $\mathbb{S}$ indicated by below lattice



Here $a_1 \neq 0, a_3 \neq 0$ but $a_1 \bullet a_3 = 0$ for $a_1, a_3 \in \mathbb{S}$ i.e. a_1 and a_3 are zero divisors of $\mathbb{S}$.

Example 2.7: Let the semiring $\mathbb{S} = Z_0{}^+ \times Z_0{}^+$, ($Z_0{}^+$ is a set of non negative integers) has zero divisors as $a_1 = (5, 0)$, $a_2 = (0, 7) \in \mathbb{S}$ and $a_1 \bullet a_2 = (0, 0)$.

Definition 2.5[11]: Let $\mathbb{S}$ be a semiring. $\mathbb{S}$ is said to be a Semidomain if in $\mathbb{S}$, $\alpha \bullet \beta = 0$ implies either $\alpha = 0$ or $\beta = 0$ for $\alpha, \beta \in \mathbb{S}$ (that is $\mathbb{S}$ has no zero divisors).

Example 2. 8 1) Z_0^+ (set of non negative integers) is a Semidomain.

2) Every Chain lattice is a Semidomain.

Definition 2.6 [9, 11, 12]: Let the semiring be $\mathbb{S}$. If there is an element $\beta \in S$ for $\alpha \in S$ such that $\alpha \bullet \beta = \beta \bullet \alpha = 1$ then in $\mathbb{S}$, α is referred to as an invertible element while element β, represented by α^{-1} , is the multiplicative inverse of α.

Note: In semiring $\mathbb{S}$, U ($\mathbb{S}$) represents the set of all invertible elements.

Definition 2.7 [11]: A commutative semiring $\mathbb{S}$ in which every nonzero element has an inverse with respect to multiplication is called a semifield.

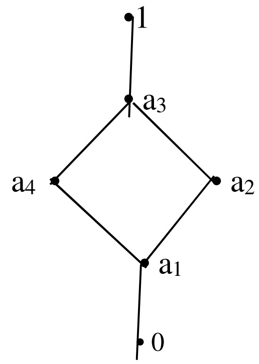
Example 2.9 [12] Q_0^+ (set of non negative rational numbers) is a semifield.

Note that every semifield is a semidomain. However, the opposite is untrue. .

Example 2.10 Z_0^+, the set containing all positive integers with zero is a Semidomain which is not a semifield.

In case of ring theory, every finite integral domain is a field but in case of semiring, a finite semidomain is not necessarily a semifield.

Example 2.11 Let $\mathbb{S}$ be a finite semidomain given by the following Lattice but it is not a field (because multiplicative inverse is absent).



Note that $a_1 \cdot a_1 = a_1 \cdot a_2 = a_1 \cdot a_3 = a_1 \cdot a_4 = a_1$. Thus a_1 has no multiplicative inverse.

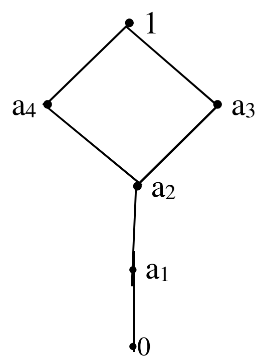
In fact every finite chain lattice other than C_2 is a finite semidoamin but it is not a semifield.

Definition 2.8 [11, 12]: Let the semiring be $\mathbb{S}$. If $\alpha + \beta = 0$ indicates that $\alpha = 0$ and $\beta = 0$ for $\alpha, \beta \in \mathbb{S}$, then $\mathbb{S}$ is referred to as a Strict Semiring.

Note that strict semiring is also called as zerosumfree semiring.

Example 2.12 [12]: Z_0^+, the set containing all positive integers with zero is a Strict Semiring.

Example 2.13: Consider a semiring S given by the following lattice



Here S is a strict semiring.

Definition 2.9: Let the semiring be $\mathbb{S}$. If

i) $\alpha + \beta = 0$ indicates that $\alpha = 0$ and $\beta = 0$ for $\alpha, \beta \in \mathbb{S}$ and

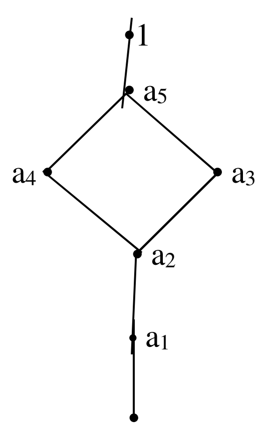
ii) $\alpha \bullet \beta = 0$ indicates that $\alpha = 0$ or $\beta = 0$ for $\alpha, \beta \in \mathbb{S}$

then $\mathbb{S}$ is said to be a Strict Semidomain.

Example 2.14 1) Z_0^+, the set containing all positive integers with zero is a strict semidomain.

2) Q_0^+, the set containing all positive rational numbers with zero is a Strict Semidomain.

3) Consider a semiring S indicated by the below lattice



Then S is a strict semidomain.

Note that every Chain lattice is a Strict Semidomain.

Definition 2.10: [12]: Let the two semirings be $\mathbb{S}$ and $\mathbb{S}'$. Semiring homomorphism refers to the mapping ψ**:** $\mathbb{S} \rightarrow \mathbb{S}'$ if $\psi(\alpha+\beta) = \psi(\alpha) + \psi(\beta)$ and $\psi(\alpha \bullet \beta) = \psi(\alpha) \bullet \psi(\beta)$ for every $\alpha, \beta \in \mathbb{S}$.

ψ is a semiring isomorphism if ψ is one one and onto map.

Definition 2.11 [11, 12]: Let the strict semiring be $\mathbb{S}$. The function $\mathbb{S} \times \mathbb{M} \rightarrow \mathbb{M}$ represented by (s, m) $\rightarrow$ s m, is known as scalar multiplication and it is commutative monoid ($\mathbb{M}$, +) with additive identity $0_{\mathbb{M}}$ for which which we have the following requirements for all elements s, s'$\in \mathbb{S}$ and all elements m, m' $\in \mathbb{M}$

1) (s s') m = s (s' m)

2) s(m + m') = s m + s m'

3) (s + s') m = s m + s' m

4) 1_S m = m

5) s $0_{\mathbb{M}} = 0_{\mathbb{M}} = 0_s$ m

Example1 2.15: 1) Let $\mathbb{S}$ be a semiring. The cartesian product $\mathbb{S} \times \mathbb{S} \times \ldots \times \mathbb{S}$ (n times) = $\mathbb{S}^n$ such that $(x_1, x_2, \ldots x_n)$, $(y_1, y_2 \ldots, y_n)$ in $\mathbb{S}^n$, $\alpha \in \mathbb{S}$ under the addition $(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$.

and scalar multiplication $\alpha(x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$ is a semimodule over $\mathbb{S}$.

2) Let $\mathbb{M} = Z_0^+ \times Z_0^+ \times \ldots \times Z_0^+$ (m times) then $\mathbb{M}$ is a Z_0^+ -Semimodule (Z_0^+ is a set of non negative integers).

3) Let C_n be a chain Lattice then $\mathbb{M} = C_n \times C_n \times \ldots \times C_n$ (m times) is a semimodule over C_n and $\mathbb{M}$ is denoted by C_n^m .

Definition 2.12[11]: If and only if $\mathbb{N}$ is closed under addition and scalar multiplication and $\mathbb{N}$ contains additive identity 0 then a non-empty subset $\mathbb{N}$ of a $\mathbb{S}$ -Semimodule $\mathbb{M}$ is a subsemimodule of $\mathbb{M}$.

Definition 2.13 [11]: If for m, m′ $\in \mathbb{M}$ with m + m′ $\in \mathbb{N}$ and m $\in \mathbb{N}$ then m′ $\in \mathbb{N}$ is subtractive subsemimodule of a $\mathbb{S}-$ Semimodule $\mathbb{M}$.

Definition 2.14 [11]: If $\mathbb{M}$ and $\mathbb{N}$ are $\mathbb{S}$-Semimodules over a strict semiring $\mathbb{S}$ then a map $\emptyset : \mathbb{M} \rightarrow \mathbb{N}$ is an $\mathbb{S}$-homomorphism if the following constraints are fulfilled.

i) For every m, m′ $\in \mathbb{M}$, $\emptyset$ (m + m′) = $\emptyset$ (m) + $\emptyset$ (m′).

ii) For every m $\in \mathbb{M}$ and every s $\in \mathbb{S}$, $\emptyset$ (sm) = s$\emptyset$ (m).

iii) $\emptyset$ (0) = 0.

Definition 2.15: The set {m$\in \mathbb{M}$/ $\emptyset$ (m) = 0} is called kernel of $\emptyset$ (ker$\emptyset$) and the set { $\emptyset$ (m) /m$\in \mathbb{M}$} is called range of $\emptyset$ where $\mathbb{M}$, $\mathbb{N}$ are $\mathbb{S}$-semimodules and $\emptyset : \mathbb{M} \rightarrow \mathbb{N}$ is an $\mathbb{S}$- homomorphism.

Note that ker$\emptyset$ and range of $\emptyset$ are subsemimodules of $\mathbb{M}$ and $\mathbb{N}$ respectively.

Theorem 2.2: ker$\emptyset$ is subtractive subsemimodule of $\mathbb{M}$.

Proof: Let m$\in$ ker$\emptyset$ and n $\in \mathbb{M}$ such that m + n $\in$ ker$\emptyset$.

Then $\emptyset$(m) = 0, $\emptyset$(m + n) = 0 = $\emptyset$(m) + $\emptyset$(n)

Therefore $\emptyset$(n) = 0.

Hence, n $\in$ ker$\emptyset$**.**

Definition 2.16[9]: The semimodule $\mathbb{M}$ over strict semiring $\mathbb{S}$ is called an $\mathbb{S}$-semilinear space.

A subspace of an $\mathbb{S}$- semilinear space is the subsemimodule of semimodule $\mathbb{M}$ over strict semiring $\mathbb{S}$.

Unless otherwise mentioned semiring $\mathbb{S}$ is assumed to be semidomain.

Note 2.2: Semilinear space elements are called vectors.

Definition 2.17[9, 12]: A subset A of a semilinear space $\mathbb{M}$ over a strict semiring $\mathbb{S}$ is said to form spanning set of $\mathbb{M}$, if every V in $\mathbb{M}$ is a linear combination of vectors $V_1, V_2, \dots, V_m \in A$ that is there exists scalars $a_1, a_2, \dots, a_m$ in $\mathbb{S}$ such that $V = a_1V_1 + a_2V_2 + \dots + a_mV_m$.

Note 2.3: Spanning set of a semilinear space $\mathbb{M}$ is also called generating set of $\mathbb{M}$. $\mathbb{M}$ is called finitely generated semilinear space if it has finite spanning set.

Definition 2.18[9, 12]: The vectors of a semilinear space $\mathbb{M}$ over strict semiring $\mathbb{S}$ are called linearly independent, if none of them can be represented as linear combination of the remaining others. Otherwise these vectors are linearly dependent.

Definition 2.19[9, 12]: Let A be a linearly independent subset of a semilinear space $\mathbb{M}$ over the strict semiring $\mathbb{S}$. A is referred to as a basis of the semilinear space $\mathbb{M}$ if it spans the semilinear space $\mathbb{M}$.

For a semiring $\mathbb{S}$ we have defined $\mathbb{S}^n$ as follows

$\mathbb{S}^n = \mathbb{S} \times \mathbb{S} \times \dots \times \mathbb{S}$ (n times) is a semilinear space over strict semiring $\mathbb{S}$, under the following addition and scalar multiplication:

For $X = (x_1, x_2, \dots, x_n)$, $Y = (y_1, y_2, \dots, y_n) \in \mathbb{S}^n$ and $\alpha \in \mathbb{S}$,

$X + Y = (x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$.

and $\alpha X = \alpha (x_1, x_2, \dots, x_n) = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$.

Proposition 2.1: Let $\mathbb{S}$ be a strict semidomain. If $X, Y \in \mathbb{S}^n$ and $X + Y = 0$ then $X = 0, Y = 0$.

Proof: Let $X = (x_1, x_2, \dots, x_n), Y = (y_1, y_2, \dots, y_n) \in \mathbb{S}^n$

$$\begin{aligned} \therefore\ X + Y &= (x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) \\ &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \\ &= (0, 0, \dots, 0) \end{aligned}$$

Therefore $x_i + y_i = 0$, for all $i = 1, 2, \dots, n$ implies $x_i = 0$, $y_i = 0$ for all i and hence, $X = 0, Y = 0$.

Also if $V_1, V_2, \dots, V_k \in \mathbb{S}^n$ then $\sum_{i=1}^{k} \alpha_i V_i = 0$ implies that $\alpha_i V_i = 0$ for all i.

Therefore $\alpha_i = 0$ or $V_i = 0$.

Note 2.4[9, 12]: In $\mathbb{S}$-semilinear space, the number of elements in each basis may vary.

Definition 2.20[9]: A free $\mathbb{S}$-semilinear space is an $\mathbb{S}$-semilinear space which has a basis over $\mathbb{S}$, where $\mathbb{S}$ is a strict semiring.

Definition 2.21[9]: In an $\mathbb{S}$-Semilinear space $\mathbb{S}^n$, two elements $V_1 = (v_{11}, v_{12}, \dots, v_{1n})$, $V_2 = (v_{21}, v_{22}, \dots, v_{2n}) \in \mathbb{S}^n$ are said to be orthogonal if $V_1 V_2^T = \begin{bmatrix} v_{11} & v_{12} & \dots & v_{1n} \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \end{bmatrix} = 0$ and $V_1 V_1^T, V_2 V_2^T \in U(\mathbb{S})$.

Let $V_1, V_2, \dots, V_m \in \mathbb{S}^n$. If $V_i V_j^T = 0$ for all $i \neq j$ and $V_i V_i^T \in U(\mathbb{S})$, $1 \le i \le n$, $1 \le j \le n$ then the set $\{V_1, V_2, \dots, V_m\}$ is said to be orthogonal.

Theorem 2.3[9]: An orthogonal subset of an $\mathbb{S}$-semilinear space $\mathbb{S}^n$ over strict semiring $\mathbb{S}$ is linearly independent.

Note 2.5 The set $\{e_1, e_2, \dots, e_n\}$, where $e_1 = (1, 0, \dots, 0_n)$, $e_2 = (0, 1, \dots, 0_n)$, $\dots, e_n = (0, 0, \dots, 1_n)$ is an orthogonal subset of $\mathbb{S}^n$. Therefore the set $\{e_1, e_2, \dots, e_n\}$ is linearly independent (by theorem 2.3). Also $\{e_1, e_2, \dots, e_n\}$ spans $\mathbb{S}^n$ and hence it is a basis of $\mathbb{S}^n$. $\{e_1, e_2, \dots, e_n\}$ is called as standard orthogonal basis of $\mathbb{S}^n$.

Definition 2.22[9]: If s = $\alpha + \beta$ implies that s = α or s = β, for all $\alpha, \beta \in \mathbb{S}$ then s $\in \mathbb{S}$ is called an additive irreducible element of $\mathbb{S}$, where $\mathbb{S}$ is a strict semiring.

Theorem 2.4[9]: In $\mathbb{S}^n$, each basis has the same number of elements if and only if 1 is an additive irreducible element, where $\mathbb{S}^n$ is a semilinear space over strict semiring $\mathbb{S}$.

Corollary 2.2[9]: 1 being additive irreducible element in strict semiring $\mathbb{S}$, $\{X_1, X_2, \dots, X_n\}$ is a basis of $\mathbb{S}$-Semilinear space $\mathbb{S}^n$ if and only if, $X_1 = (x_{11}, 0, \dots, 0)$, $X_2 = (0, x_{22}, \dots, 0)$, … , $X_n = (0, 0, \dots, x_{nn})$, with $x_{ii} \in U(\mathbb{S})$ for any i, $1 \leq i \leq n$

Corollary 2.3[9]: If U ($\mathbb{S}$) = {1} then {$e_1, e_2, \dots, e_n$} is the unique basis of $\mathbb{S}^n$, where $\mathbb{S}^n$ be a semilinear space over strict semiring $\mathbb{S}$ with 1 as an additive irreducible element.

Note that in corollary 2.2 for all $x_{ii} \in U(\mathbb{S}) = \{1\}$ means $x_{ii} = 1$. Therefore each $X_i = e_i$, $1 \leq i \leq n$.

Definition 2.23[9]: If any two bases of a free semilinear space $\mathbb{M}$ over strict semiring $\mathbb{S}$ have the same number of elements then the number elements in a basis is the dimension of $\mathbb{M}$ over $\mathbb{S}$ and is represented by $dim_{\mathbb{S}}\, \mathbb{M}$.

Note that any two bases of $\mathbb{S}^n$ have the same number of elements only when $\mathbb{S}$ is a strict semiring with 1 as an additive irreducible element. Therefore, $dim_{\mathbb{S}}(\mathbb{S}^n) = n$.

Theorem 2.5[9]: A set $\{X_1, X_2, \dots, X_n\}$ is a basis of $\mathbb{S}^n$ if and only if it is orthogonal, where $\mathbb{S}^n$ be n-dimensional Semilinear space over a strict semiring

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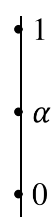
Corollary 2.4[9]: In $\mathbb{S}^n$, the number of elements in every orthogonal set is not more than n where $\mathbb{S}^n$ is n-dimensional $\mathbb{S}$-Semilinear space.

Note 2.6: As $\mathbb{S}$ is finite strict semidomain, $\mathbb{S}^n$ and every semilinear subspace $\mathbb{M}$ of $\mathbb{S}^n$ is finite. Thus $\mathbb{M}$ has finite generating set which implies it has a basis. Therefore every semilinear subspace of $\mathbb{S}^n$ is free.

Example 2.16: Chain Lattice C_3 indicated as below is a Strict Semiring (Every Chain lattice is a Strict Semiring) .

$C_3 = \{0, \alpha, 1\}$



Semilinear space over C_3:

C_3^3 = { (0, 0, 0) , (0, α, 1), (0, 1, α), (α, α, 1), (α, 1, α), (1, α, 1), (1, 1, α), (α, 1, 1), (1, α, α), (1, 0, α), (1, 1, 0), (0, 1, 1), (1, 1, α) (0, α, 0), (0, 0, α), (α, 0, 0), (α, α, 0), (α, 0, α), (0, α, α), (α, α, α), (0, α, 1), (α, 1, 0), (α, 0, 1), (1, α, 0), (1, 1, 1), (1, 0, 0), (1, 0, 1)}.

Basis of C_3^3 : {(1, 0, 0), (0, 1, 0), (0, 0, 1)**}.**

Dimension of C_3^3 : dim C_3^3 = 3.

Subspaces of Semilinear space C_3^3 over C_3:

U_1 = {(0, 0, 0), (α, 0, 0), (1, 0, 0)}.

Basis of U_1: {(1, 0, 0)**}.**

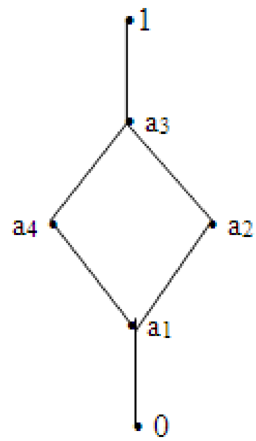
Dimension of U_1: dim U_1= 1.

U_2 = {(0, 0, 0), (0, 0, 1), (1, 0, α), (α, 0, 1), (1, 0, 1), (0, 0, α), (α, 0, 0), (α, 0, α), (1, 0, 0)}.

Basis of U_2: {(1, 0, 0),), (0, 0, 1)**}.**

Dimension of U_2: dim U_2= 2.

Example 2.17: Let S be a Strict Semiring given by the following lattice.



Here $S = \{0, a_1, a_2, a_3, a_4, 1\}$.

Semilinear space over S:

$S^5 = \{(x_1, x_2, x_3, x_4, x_5)/x_1, x_2, x_3, x_4, x_5 \in S\}$.

Basis of S^5: $\{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)\}$.

Dimension of S^5: dim $S^5 = 5$.

Subspaces of Semilinear space S^5 over S:

i) $T_1 = \{(p, p, q, 0, t)/p, q, t \in S\}$ is the subspace of Semilinear space S^5 over S.

$T_1 = \langle (1, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 0, 1) \rangle$.

Also $\{(1, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 0, 1)\}$ is an orthogonal set and hence basis of T_1.

Therefore, dimension of T_1 is 3. That is, $dimT_1 = 3$.

ii) $T_2 = \{(0, l, m, n, 0)/l, m, n \in S\}$ is the subspace of Semilinear space S^5 over S.

Also $\langle 0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0) \rangle$ is an orthogonal set and hence basis of T_2.

Therefore, dimension of T_2 is 3. That is, $dimT_2 = 3$.

Definition 2.24[14]: T: $\mathbb{S}^n \rightarrow \mathbb{S}^m$ be a linear transformation which fulfills the constraints

i) $T(X_1 + X_2) = T(X_1) + T(X_2)$, $X_1, X_2 \in \mathbb{S}^n$

ii) $T(aX) = a\,T(X)$, $a \in \mathbb{S}$, $X \in \mathbb{S}^n$

Constraints (i) and (ii) are equal to just one constraint

$T(aX_1+bX_2) = a\,T(X_1) + b\,T(X_2)$, $a, b \in \mathbb{S}$, $X_1, X_2 \in \mathbb{S}^n$

Definition 2.25[14]: T: $\mathbb{S}^n \rightarrow \mathbb{S}^m$ be a linear transformation for which kernel is the set of all vectors $X \in \mathbb{S}^n$ such that $T(X) = 0$.

Definition 2.26[14]: T: $\mathbb{S}^n \rightarrow \mathbb{S}^m$ be a linear transformation for which range is the set of all vectors $X' \in \mathbb{S}^m$ such that $T(X) = X'$ where $X \in \mathbb{S}^n$.

Theorem 2.6[14] If T: $\mathbb{S}^n \rightarrow \mathbb{S}^m$ be a linear transformation then kernel and range of T are semilinear subspaces of $\mathbb{S}^n$.

Proposition 2.2 Let $\mathbb{S}^n$ be a finite dimensional semilinear space over a strict semiring $\mathbb{S}$, T: $\mathbb{S}^n \rightarrow \mathbb{S}^n$ be a linear transformation and $U + V \in$ kerT then $U, V \in$ kerT.

Proof: Let $U + V \in$ kerT. It means $T(U + V) = 0$

But T (U + V) = T (U) + T (V). Therefore, T (U) + T (V) = 0 which implies that T (U) = 0, T(V) = 0 (By proposition 2.1)

Hence, U, V ∈ kerT

Thus, U + V ∈ kerT implies that U, V ∈ kerT.

Proposition 2.3: Let T: $\mathbb{S}^n \rightarrow \mathbb{S}^n$ be a linear transformation and $e_1, e_2, \ldots, e_n$ be a standard basis of $\mathbb{S}^n$ then kerT = $\mathbb{S}^k$, $k \leq n$.

Proof: Let X ∈ kerT implies that X ∈ $\mathbb{S}^n$ which implies that T (X) = 0.

Take $X = \sum_{i=1}^{n} \alpha_i e_i$.

ie. $T(X) = \sum_{i=1}^{n} \alpha_i T(e_i) = 0$.

Therefore $\alpha_i = 0$ or T (e_i) = 0 for all i.

If $\alpha_i \neq 0$ then e_i ∈ kerT

Therefore for all X ∈ kerT, all e_i involved in linear combination of X belongs to kerT

Hence we can take kerT = $\langle e_1, e_2, \ldots, e_k \rangle$

That is kerT = $\mathbb{S}^k$, $k \leq n$.

Definition 2.27[14]: The dimension of range of a linear transformation T is referred to as a rank and it is represented by Rank (T). Also the dimension of kernel of a linear transformation T is referred to as a nullity and represented by Nullity (T).

Hamming Weight: Let X ∈ $\mathbb{S}^n$, X = ($x_1, x_2, \ldots, x_n$) then w (X) = Number of non-zero x_i, $1 \leq i \leq n$.

Example 2.28: Let X ∈ $\mathbb{S}^5$, X = (2, 0, 3, 0, 1) then w(X) = 3.

Rank-Nullity theorem in semilinear spaces over strict semidomain

Theorem 3.1 Let S be a finite strict semidomain and T: $S^n \rightarrow S^n$ be a linear transformation such that $\langle T(X), T(Y) \rangle = 0$, if $\langle X, Y \rangle = 0$ for X, Y ∈ S^n. Then Rank (T) + Nullity (T) = n.

Proof: Let {$e_1, e_2, \ldots, e_n$} be a standard orthogonal basis of S^n.

Case (i) Let T (e_i) = 0 for all i.

Then ker (T) = S^n and Im (T) = 0.

Therefore, Rank (T) = 0 and Nullity (T) = n.

Thus, Rank (T) + Nullity (T) = n.

Case (ii) Let T (e_i) ≠ 0 for all i.

Then {T (e_1), T(e_2), . . . , T(e_n)} is orthogonal subset of S^n as $\langle e_i, e_j \rangle = 0$

Implies $\langle T(e_i), T(e_j) \rangle = 0$ for $i \neq j$.

Hence orthogonal basis of ImT = $\langle T(e_1), T(e_2), \ldots, T(e_n) \rangle$

Therefore, Rank (T) = n.

In this case, ker (T) = 0 and hence Nullity (T) = 0.

Thus, Rank (T) + Nullity (T) = n.

Case (iii) Let $T(e_i) \neq 0$ for $1 \leq i \leq k$.

$T(e_i) = 0$ for $k + 1 \leq i \leq n$.

Then ImT = $\langle T(e_1), T(e_2), \ldots, T(e_k) \rangle$

As in case (ii) {T (e_1), T (e_2), . . . , T (e_n)} is an orthogonal basis of Im(T).

Hence Rank (T) = k.

Now T (e_i) = 0 for $k + 1 \leq i \leq n$ giving that $\langle e_{k+1}, e_{k+2}, \ldots, e_n \rangle \subset \ker(T)$.

Let X ∈ ker (T) which implies that $X = \sum_{i=1}^{n} x_i e_i$.

Therefore $T(X) = \sum_{i=1}^{n} x_i T(e_i) = 0$.

Thus $x_i\, T(e_i) = 0$ for all i, $1 \le i \le n$.

For $1 \le i \le k$, $T(e_i) \neq 0$ giving that $x_i = 0$, $1 \le i \le k$.

Thus, Nullity (T) = n – k.

Hence Rank (T) + Nullity (T) = k + n – k = n.

Theorem 3.2 Let $\mathbb{S}$ be a finite strict semidomain, $1 \in \mathbb{S}$ is an additive irreducible element, 1 + 1 = 1, U ($\mathbb{S}$) = 1 and T: $\mathbb{S}^n \longrightarrow \mathbb{S}^n$ be a linear transformation. If $V_1, \ldots, V_k$ is an orthogonal basis of range of T with w (V_i) = w_i , $w_1 + w_2 + \ldots + w_k = m$ then Rank (T) + Nullity (T) = k + n - m.

Proof: Let $\{e_1, e_2, \ldots, e_n\}$ be a standard orthogonal basis of $\mathbb{S}^n$.

Now $\{V_1, \ldots, V_k\}$ is an orthogonal basis of range of T and hence without loss of generality we can assume

$V_1 = \left(1, 1, \ldots, 1_{w_1}, 0, \ldots, 0\right)$

$V_2 = \left(0, 0, \ldots, 0_{w_1}, 1, 1, \ldots, 1_{w_2}, 0, \ldots, 0\right)$

$\vdots$

$V_k = \left(0, 0, \ldots, 0_{w_1}, 0, 0, \ldots, 0_{w_2}, \ldots, 0, 0, \ldots, 0_{w_{k-1}}, 1, 1, \ldots, 1_{w_k}, 0, \ldots, 0\right)$

Let M = $\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_k \end{bmatrix}$ and X = $(x_1, x_2, \ldots, x_n) \in \ker T$.

$Therefore,$ T (X) = $MX^T = 0$

It means T (X) = $\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0$

$$T(X) = \begin{bmatrix} 1 & 1 & \cdots & 1w_1 & 0 & 0 & \cdots & 0w_2 & \cdots & 0 & 0 & \cdots & 0w_k & \cdots & 0 \\ 0 & 0 & \cdots & 0w_1 & 1 & 1 & \cdots & 1w_2 & \cdots & 0 & 0 & \cdots & 0w_k & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0w_1 & 0 & 0 & \cdots & 0w_2 & \cdots & 1 & 1 & \cdots & 1w_k & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Thus, T (X) = $\begin{bmatrix} x_1 + x_2 + \ldots + x_{w_1} \\ x_{w_1+1} + \ldots + x_{w_1+w_2} \\ \vdots \\ x_{w_1+w_2+\ldots+w_{k-1}+1} + \ldots + x_{w_1+w_2+\ldots+w_k} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Which implies that $x_1 + x_2 + \ldots + x_{w_1} = 0$,

$x_{w_1+1} + \ldots + x_{w_1+w_2} = 0$,

$$x_{w_1+w_2+\ldots+w_{k-1}+1} + \ldots + x_{w_1+w_2+\ldots+w_k} = 0$$

Since $\mathbb{S}$ is a strict semiring, hence

$$x_1 = x_2 = \ldots = x_{w_1} = x_{w_1} + x_{w_2} = x_{w_1+w_2+\ldots+w_{k-1}+1} = \ldots = x_{w_1+w_2+\ldots+w_k} = 0$$

But $w_1 + w_2 + \ldots + w_k = m$

Therefore, $x_1 = x_2 = \ldots = x_m = 0$.

Thus, $X = (0,\ldots, 0_m, x_{m+1},\ldots,x_n)$.

Hence $X \in \ker T$ implies that $X = \sum_{i=m+1}^{n} x_i e_i$

Also each $e_i \in \ker T$ for $m + 1 \leq i \leq n$

Therefore, $kerT = \langle e_{m+1}, e_{m+2}, \ldots, e_n \rangle$

Thus, $\{e_{m+1}, e_{m+2}, \ldots, e_n\}$ is a basis of kerT.

Hence dim (kerT) = n - m

Therefore, dim (Range T) + dim (ker T) = k + n – m.

That is Rank (T) + Nulity (T) = k + n – m (by definition 3.12).

Example 3.1 Let $T: S^5 \to S^5$ be a linear transformation on semilinear space S^5 defined as $T(a_1, a_2, a_3, a_4, a_5) = (a_1, a_1, a_1, a_2, 0)$.

Then we can write $T(a_1, a_2, a_3, a_4, a_5) = (a_1, a_1, a_1, a_2, 0) = a_1(1, 1, 1, 0, 0) + a_2(0, 0, 0, 1, 0)$.

Denote $V_1 = (1, 1, 1, 0, 0)$ and $V_2 = (0, 0, 0, 1, 0)$. Here $V_1 V_2^T = 0$.

Therefore, the set $\{V_1, V_2\}$ is orthogonal set. Hence it is linearly independent.

Also the set $\{V_1, V_2\}$ spans range of T. Therefore, it is a basis of range T.

Thus, $dim\,(RangeT) = 2$.

Here $w(V_1) = 3$ and $w(V_2) = 1$. Therefore, $m = w(V_1) + w(V_2) = 4$.

Let $M = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ and $B = (b_1, b_2, b_3, b_4, b_5) \in kerT$.

Therefore, $T(B) = MB^T = 0$.

Implies that $\begin{bmatrix} b_1 + b_2 + b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Therefore, $b_1 + b_2 + b_3 = 0$, $b_4 = 0$.

Since S is a strict semiring. It implies that $b_1 = b_2 = b_3 = b_4 = 0$.

Thus, $X = (0, 0, 0, 0, x_5)$. Hence $kerT = \langle (0, 0, 0, 0, 1) \rangle$.

Therefore, $dim\,(kerT) = 1$.

Thus, $dim\,(RangeT) + dim(kerT) = 2 + 1 = 3$ and $k + n - m = 2 + 5 - 4 = 3$.

Hence the result.

Corollary 3.1 Let $\mathbb{S}$ be a finite strict semidomain, $\mathbb{S}^n$ be a finite dimensional $\mathbb{S}$-semilinear space, $T: \mathbb{S}^n \longrightarrow \mathbb{S}^n$ be a linear transformation. If $\{V_1, V_2, \ldots, V_k\}$ is an orthogonal basis of RangeT with $w(V_i) = 1$, for all i then Range T $= \mathbb{S}^k$ and Rank (T) + Nullity (T) = n.

Proof: Let $\mathbb{S}^n$ be a finite dimensional $\mathbb{S}$-semilinear space and $\{e_1, e_2,\ldots,e_n\}$ be a standard orthogonal basis of $\mathbb{S}^n$. Also $T: \mathbb{S}^n \longrightarrow \mathbb{S}^n$ be a linear transformation and range of T is a semilinear subspace of $\mathbb{S}$n (by theorem 2.6).

$\{V_1, V_2\ldots, V_k\}$ is an orthogonal basis of RangeT and $w(V_i) = 1$, for all i.

Then $w_1 + w_2 + \cdots + w_k = k.$

Hence Range T = $\mathbb{S}^k$.

Thus, dim (RangeT) + dim (kerT) = k + n – k = n.

That is Rank (T) + Nullity (T) = k + n – k = n (by definition 2.27).

Conclusions: In this contribution we have identified certain semirings and proved Rank-Nullity theorem in semilinear spaces over special semiring called strict semiring.

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