On Neutrosophic Soft Generalized Closed Sets

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Abstract

This study focuses on the theoretical development of neutrosophic soft generalized closed sets, building upon concepts previously introduced in the literature. We provide formal definitions, examine fundamental properties, and establish several key theorems that characterize these sets. The main structural and topological features arising from the combination of neutrosophic soft sets and generalized sets are analyzed rigorously. This work aims to deepen the understanding of neutrosophic soft generalized closed sets within topological frameworks and to lay a solid foundation for future theoretical research in this area.

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1 Introduction

Since the beginning of human existence, individuals have faced numerous uncertainties in various aspects of life. Classical set theory, introduced by Cantor [7], is often insufficient in addressing such uncertainties. To model indeterminacy in disciplines such as mathematics, engineering, social sciences, and topology, several alternative set theories have been developed. Among these are fuzzy sets proposed by Zadeh [32], intuitionistic fuzzy sets introduced by Atanassov [4], soft sets defined by Molodtsov [19], and neutrosophic sets formulated by Smarandache [28].

In recent decades, researchers have not only studied these individual theories but also explored hybrid structures arising from their interaction. One notable extension is neutrosophic soft set theory, initially proposed by Maji [17] and further developed by Deli and Broumi [9]. Subsequent studies introduced variations and operations on neutrosophic soft sets, including the melodic neutrosophic soft set [6], and explored their topological properties [5, 20, 21]. These structures have been applied in diverse areas such as decision-making, game theory, algebra, and medical diagnosis [1–3, 8, 11, 22, 33].

Closed sets are fundamental in topology, forming the basis for topological structures through either Kuratowski's closure axioms or the axioms of closed sets. Levine [15] extended this notion by introducing generalized closed sets (g-closed sets), which not only generalize classical closed sets but also yield new topological properties. This extension led to weaker separation axioms such as $T_{1/2}$, T_{gs}, and $T_{3/4}$, enriching the theory of topological spaces [10, 12–14, 16, 18, 24, 25].

Building on these foundations, researchers combined generalized set theory with neutrosophic and soft set frameworks, leading to generalized neutrosophic soft sets and their associated properties [6, 26, 27, 29, 31].

The main objective of this paper is to advance the theoretical development of neutrosophic soft generalized closed sets, as introduced by Özkan and Yazgan [23]. We present formal definitions, explore fundamental properties, and establish key theorems. Finally, we analyze new set structures arising from the combination of neutrosophic soft and generalized sets, examining their interrelations and characteristics through rigorous proofs and illustrative examples.

2 Preliminaries

This section briefly introduces the key concepts, definitions, and theorems related to neutrosophic soft sets, neutrodophic soft topological spaces, and generalized sets. The fundamental properties of neutrodophic soft closed sets and generalized closed sets used in this study are also discussed and illustrated with examples.

Definition 2.1 (Soft Set [19])**.** *Let Ω be a universal set, Λ a set of parameters, and $\wp(\Omega)$ the family of all subsets of Ω. If there exists a mapping $\Psi : A \to \wp(\Omega)$ with $A \subseteq \Lambda$, then the pair (Ψ, A) is called a* soft set *over Ω with respect to Λ. In this context, a soft set (Ψ, A) can be regarded as a parameterized family of subsets of Ω. For each $\lambda \in A$, $\Psi(\lambda)$ represents the collection of λ-elements, and hence the soft set is represented by*

$$(\Psi, A) = \{(\lambda, \Psi(\lambda)) : \lambda \in A,\ \Psi : A \to \wp(\Omega)\}.$$

Definition 2.2 (Neutrosophic Set [28])**.** *On the universal set Ω, a* neutrosophic set *M is given by*

$$M = \{\langle \nu, T_M(\nu), I_M(\nu), F_M(\nu)\rangle : \nu \in \Omega\},$$

where

$$T_M, I_M, F_M : \Omega \to [0, 1] \quad \text{and} \quad 0 \le T_M(\nu) + I_M(\nu) + F_M(\nu) \le 3.$$

Here, $T_M(\nu)$, $I_M(\nu)$ and $F_M(\nu)$ denote, respectively, the truth-membership, indeterminacy-membership, and falsity-membership degrees of ν with respect to M. Although neutrosophic sets are originally defined over the extended interval $]-0, 1^+[$, in most *applied sciences and engineering problems they are usually restricted to the standard interval* $[0, 1]$ *for practicality.*

The concept of a *neutrosophic soft set*, first proposed by Maji (2013) and later reformulated by Deli and Broumi (2015), is introduced as follows.

Definition 2.3 (Neutrosophic Soft Set [9])**.** *Let* Ω *be a universal set,* Λ *a set of parameters, and* $\wp(\Omega)$ *the collection of all neutrosophic sets on* Ω*. A* neutrosophic soft set *over* Ω *with respect to* Λ *is a pair* (Ψ, Λ)*, where* Ψ *is a mapping*

$$\Psi : \Lambda \to \wp(\Omega).$$

In other words, each parameter $\lambda \in \Lambda$ *is associated with a neutrosophic set* $\Psi(\lambda)$ *on* Ω*, and the neutrosophic soft set can be written as*

$$(\Psi, \Lambda) = \{(\lambda, \langle \nu, T_{\Psi(\lambda)}(\nu), I_{\Psi(\lambda)}(\nu), F_{\Psi(\lambda)}(\nu)\rangle) : \nu \in \Omega, \lambda \in \Lambda\}.$$

The family of all neutrosophic soft sets over (Ω, Λ) *is denoted by* $\mathrm{NSS}(\Omega_\Lambda)$*.*

Here, $T_{\Psi(\lambda)}(\nu)$*,* $I_{\Psi(\lambda)}(\nu)$ *and* $F_{\Psi(\lambda)}(\nu) \in [0, 1]$ *represent the truth, indeterminacy, and falsity membership functions, respectively, and satisfy*

$$0 \le T_{\Psi(\lambda)}(\nu) + I_{\Psi(\lambda)}(\nu) + F_{\Psi(\lambda)}(\nu) \le 3.$$

For brevity, we shall denote a neutrosophic soft set simply by Ψ_Λ *instead of* (Ψ, Λ) *throughout this work.*

Definition 2.4 (Basic Operations on Neutrosophic Soft Sets [6,17,21])**.** *Let* $\Psi_\Lambda, \Phi_\Lambda \in \mathrm{NSS}(\Omega_\Lambda)$*. The following operations are defined:*

(a) ***Complement:*** *The complement of* Ψ_Λ*, denoted by* $(\Psi_\Lambda)^c$*, is given as*

$$(\Psi_\Lambda)^c = \{(\lambda, \langle \nu, F_{\Psi(\lambda)}(\nu), 1 - I_{\Psi(\lambda)}(\nu), T_{\Psi(\lambda)}(\nu)\rangle) : \nu \in \Omega, \lambda \in \Lambda\}.$$

It satisfies $((\Psi_\Lambda)^c)^c = \Psi_\Lambda$*.*

(b) ***Neutrosophic Soft Subset:*** *We say* Ψ_Λ *is a neutrosophic soft subset of* Φ_Λ*, written* $\Psi_\Lambda \subseteq \Phi_\Lambda$*, if for every* $\lambda \in \Lambda$ *and* $\nu \in \Omega$ *we have*

$$T_{\Psi(\lambda)}(\nu) \le T_{\Phi(\lambda)}(\nu), \quad I_{\Psi(\lambda)}(\nu) \le I_{\Phi(\lambda)}(\nu), \quad F_{\Psi(\lambda)}(\nu) \ge F_{\Phi(\lambda)}(\nu).$$

Moreover, if $\Psi_\Lambda \subseteq \Phi_\Lambda$ *and* $\Phi_\Lambda \subseteq \Psi_\Lambda$*, then* $\Psi_\Lambda = \Phi_\Lambda$*.*

(c) ***Empty and Absolute Neutrosophic Soft Sets:***

- Ψ_Λ *is called* empty*, denoted* $\emptyset_\Lambda$*, if*

$$T_{\Psi(\lambda)}(\nu) = 0, \quad I_{\Psi(\lambda)}(\nu) = 0, \quad F_{\Psi(\lambda)}(\nu) = 1.$$

- Ψ_Λ *is called* absolute*, denoted* 1_Λ*, if*

$$T_{\Psi(\lambda)}(\nu) = 1, \quad I_{\Psi(\lambda)}(\nu) = 1, \quad F_{\Psi(\lambda)}(\nu) = 0.$$

- *Clearly,* $(\emptyset_\Lambda)^c = 1_\Lambda$ *and* $(1_\Lambda)^c = \emptyset_\Lambda$.

(d) ***Union:*** *The union* $\Psi_\Lambda \cup \Phi_\Lambda = \Upsilon_\Lambda$ *is defined, for each* $\lambda \in \Lambda$ *and* $\nu \in \Omega$*, by*

$$\begin{aligned} T_{\Upsilon(\lambda)}(\nu) &= \max\{T_{\Psi(\lambda)}(\nu),\, T_{\Phi(\lambda)}(\nu)\}, \\ I_{\Upsilon(\lambda)}(\nu) &= \max\{I_{\Psi(\lambda)}(\nu),\, I_{\Phi(\lambda)}(\nu)\}, \\ F_{\Upsilon(\lambda)}(\nu) &= \min\{F_{\Psi(\lambda)}(\nu),\, F_{\Phi(\lambda)}(\nu)\}. \end{aligned}$$

(e) ***Intersection:*** *The intersection* $\Psi_\Lambda \cap \Phi_\Lambda = \Upsilon_\Lambda$ *is defined by*

$$\begin{aligned} T_{\Upsilon(\lambda)}(\nu) &= \min\{T_{\Psi(\lambda)}(\nu),\, T_{\Phi(\lambda)}(\nu)\}, \\ I_{\Upsilon(\lambda)}(\nu) &= \min\{I_{\Psi(\lambda)}(\nu),\, I_{\Phi(\lambda)}(\nu)\}, \\ F_{\Upsilon(\lambda)}(\nu) &= \max\{F_{\Psi(\lambda)}(\nu),\, F_{\Phi(\lambda)}(\nu)\}. \end{aligned}$$

(f) ***Difference:*** *The difference* $\Psi_\Lambda \setminus \Phi_\Lambda = \Upsilon_\Lambda$ *is defined as*

$$\Psi_\Lambda \setminus \Phi_\Lambda = \Psi_\Lambda \cap (\Phi_\Lambda)^c,$$

which leads to

$$\begin{aligned} T_{\Upsilon(\lambda)}(\nu) &= \min\{T_{\Psi(\lambda)}(\nu),\, T_{\Phi(\lambda)}(\nu)\}, \\ I_{\Upsilon(\lambda)}(\nu) &= \min\{I_{\Psi(\lambda)}(\nu),\, 1 - I_{\Phi(\lambda)}(\nu)\}, \\ F_{\Upsilon(\lambda)}(\nu) &= \max\{F_{\Psi(\lambda)}(\nu),\, F_{\Phi(\lambda)}(\nu)\}. \end{aligned}$$

Proposition 2.1 (Properties of Union and Intersection [21])**.** *Let* $\Psi_\Lambda, \Phi_\Lambda, \Upsilon_\Lambda \in \mathrm{NSS}(\Omega_\Lambda)$*. The following identities are valid:*

(i) $\Psi_\Lambda \cup \big(\Phi_\Lambda \cup \Upsilon_\Lambda\big) = \big(\Psi_\Lambda \cup \Phi_\Lambda\big) \cup \Upsilon_\Lambda$,

(ii) $\Psi_\Lambda \cap \big(\Phi_\Lambda \cap \Upsilon_\Lambda\big) = \big(\Psi_\Lambda \cap \Phi_\Lambda\big) \cap \Upsilon_\Lambda$,

(iii) $\Psi_\Lambda \cup \big(\Phi_\Lambda \cap \Upsilon_\Lambda\big) = \big(\Psi_\Lambda \cup \Phi_\Lambda\big) \cap \big(\Psi_\Lambda \cup \Upsilon_\Lambda\big)$,

(iv) $\Psi_\Lambda \cap \big(\Phi_\Lambda \cup \Upsilon_\Lambda\big) = \big(\Psi_\Lambda \cap \Phi_\Lambda\big) \cup \big(\Psi_\Lambda \cap \Upsilon_\Lambda\big)$,

(v) $\Psi_\Lambda \cup \emptyset_\Lambda = \Psi_\Lambda, \quad \Psi_\Lambda \cap \emptyset_\Lambda = \emptyset_\Lambda$,

(vi) $\Psi_\Lambda \cup 1_\Lambda = 1_\Lambda, \quad \Psi_\Lambda \cap 1_\Lambda = \Psi_\Lambda$.

Proposition 2.2 (Complement Properties [21])**.** *Let* $\Psi_\Lambda, \Phi_\Lambda \in \mathrm{NSS}(\Omega_\Lambda)$*. Then the following equalities hold:*

(i) $(\Psi_\Lambda \cup \Phi_\Lambda)^c = \Psi_\Lambda^c \cap \Phi_\Lambda^c$,

(ii) $(\Psi_\Lambda \cap \Phi_\Lambda)^c = \Psi_\Lambda^c \cup \Phi_\Lambda^c$.

Definition 2.5 (Neutrosophic Soft Topology [21])**.** *Let* $\mathcal{T}_{\mathrm{NSS}} \subseteq \mathrm{NSS}(\Omega_\Lambda)$ *be a family of neutrosophic soft sets. If* $\mathcal{T}_{\mathrm{NSS}}$ *satisfies the following conditions, it is called a* neutrosophic soft topology *on* Ω*:*

1. $\emptyset_\Lambda, 1_\Lambda \in \mathcal{T}_{\mathrm{NSS}}$,

2. $\mathcal{T}_{\text{NSS}}$ *is closed under arbitrary unions,*

3. $\mathcal{T}_{\text{NSS}}$ *is closed under finite intersections.*

The triple $(\Omega, \mathcal{T}_{\text{NSS}}, \Lambda)$ *is then called a* neutrodophic soft topological space *(NSTS). Elements of* $\mathcal{T}_{\text{NSS}}$ *are referred to as* neutrosophic soft open sets *(NSOS). A neutrosophic soft set* Ψ_Λ *is said to be* neutrosophic soft closed *(NSCS) if its complement* Ψ_Λ^c *belongs to* $\mathcal{T}_{\text{NSS}}$.

Proposition 2.3 (Properties of NSCSs [21])**.** *Let* $(\Omega, \mathcal{T}_{\text{NSS}}, \Lambda)$ *be an NSTS. Then the following properties hold:*

(i) The neutrosophic soft sets $\emptyset_\Lambda$, 1_Λ, *and* Ω *are all closed sets.*

(ii) The intersection of any collection of NSCSs is again an NSCS.

(iii) The union of finitely many NSCSs is an NSCS.

Definition 2.6 (Discrete and Non-Discrete Neutrosophic Soft Topologies [21])**.** *(i) The family* $\mathcal{T}_{\text{NSS}} = \{\emptyset_\Lambda, 1_\Lambda\}$ *is called the* neutrosophic soft non-discrete topology *on* Ω, *and the triple* $(\Omega, \mathcal{T}_{\text{NSS}}, \Lambda)$ *is called a* neutrosophic soft non-discrete topological space.

(ii) If $\mathcal{T}_{\text{NSS}} = \text{NSS}(\Omega_\Lambda)$, *then* $\mathcal{T}_{\text{NSS}}$ *is called the* neutrosophic soft discrete topology *on* Ω, *and* $(\Omega, \mathcal{T}_{\text{NSS}}, \Lambda)$ *is called a* neutrosophic soft discrete topological space.

Definition 2.7 (Neutrosophic Soft Point [21])**.** *Let* $\text{NSS}(\Omega_\Lambda)$ *denote the family of all neutrosophic soft sets over the universe* Ω. *A neutrosophic soft set of the form*

$$\nu^\lambda_{(\alpha,\beta,\gamma)}$$

is called a neutrosophic soft point *for each* $\nu \in \Omega$, $\lambda \in \Lambda$, *and* $0 < \alpha, \beta, \gamma \leq 1$, *defined by*

$$\nu^\lambda_{(\alpha,\beta,\gamma)}(\lambda')(\mu) = \begin{cases} (\alpha, \beta, \gamma), & \text{if } \lambda' = \lambda \text{ and } \mu = \nu, \\ (0, 0, 1), & \text{otherwise.} \end{cases}$$

Definition 2.8 (Fundamental Properties of Neutrosophic Soft Points [21])**.** *Let* $\nu^\lambda_{(\alpha,\beta,\gamma)}$ *and* $\mu^{\lambda'}_{(\alpha',\beta',\gamma')}$ *be neutrosophic soft points in* Ω_Λ. *Then:*

(a) ***Equality:***

$$\nu^\lambda_{(\alpha,\beta,\gamma)} = \mu^{\lambda'}_{(\alpha',\beta',\gamma')} \quad \text{iff} \quad \nu = \mu,\ \lambda = \lambda',\ (\alpha, \beta, \gamma) = (\alpha', \beta', \gamma').$$

(b) ***Membership:*** *For a neutrosophic soft set* Ψ_Λ,

$$\nu^\lambda_{(\alpha,\beta,\gamma)} \in \Psi_\Lambda \quad \text{iff} \quad \Psi_\Lambda(\lambda)(\nu) = (\alpha', \beta', \gamma') \text{ with } \alpha \leq \alpha',\ \beta \geq \beta',\ \gamma \geq \gamma'.$$

(c) ***Inclusion:***

$$\nu^\lambda_{(\alpha,\beta,\gamma)} \subseteq \nu^\lambda_{(\alpha',\beta',\gamma')} \quad \text{iff} \quad \alpha \leq \alpha',\ \beta \geq \beta',\ \gamma \geq \gamma'.$$

(d) ***Support:*** *The support of* $\nu^{\lambda}_{(\alpha,\beta,\gamma)}$ *is*

$$\mathrm{supp}(\nu^{\lambda}_{(\alpha,\beta,\gamma)}) = \{\nu \in \Omega \mid \alpha > 0,\ \beta < 1,\ \gamma < 1\}.$$

Definition 2.9 (Interior of a Neutrosophic Soft Set [21]). *Let* $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$ *be an NSTS and* $\Psi_\Lambda \in \mathrm{NSS}(\Omega_\Lambda)$. *The* interior *of* Ψ_Λ, *denoted by* $\mathrm{NsInt}(\Psi_\Lambda)$, *is given by*

$$\mathrm{NsInt}(\Psi_\Lambda) = \bigcup\{\Phi_\Lambda \in \mathcal{T}_{\mathrm{NSS}} \mid \Phi_\Lambda \subseteq \Psi_\Lambda\}.$$

In other words, the interior is the largest neutrosophic soft open set contained in Ψ_Λ.

Theorem 2.1 (Properties of Neutrosophic Soft Interior [21]). *Let* $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$ *be an NSTS, and let* $\Psi_\Lambda, \Phi_\Lambda \in \mathrm{NSS}(\Omega_\Lambda)$. *Then:*

(i) $\mathrm{NsInt}(\Psi_\Lambda) \subseteq \Psi_\Lambda$, *and it is the largest NSOS contained in* Ψ_Λ.

(ii) If $\Psi_\Lambda \subseteq \Phi_\Lambda$, *then* $\mathrm{NsInt}(\Psi_\Lambda) \subseteq \mathrm{NsInt}(\Phi_\Lambda)$.

(iii) $\mathrm{NsInt}(\Psi_\Lambda)$ *is a neutrosophic soft open set:* $\mathrm{NsInt}(\Psi_\Lambda) \in \mathcal{T}_{\mathrm{NSS}}$.

(iv) Ψ_Λ *is neutrosophic soft open iff* $\mathrm{NsInt}(\Psi_\Lambda) = \Psi_\Lambda$.

(v) $\mathrm{NsInt}(\mathrm{NsInt}(\Psi_\Lambda)) = \mathrm{NsInt}(\Psi_\Lambda)$.

(vi) $\mathrm{NsInt}(\emptyset_\Lambda) = \emptyset_\Lambda$ *and* $\mathrm{NsInt}(1_\Lambda) = 1_\Lambda$.

(vii) $\mathrm{NsInt}(\Psi_\Lambda \cap \Phi_\Lambda) = \mathrm{NsInt}(\Psi_\Lambda) \cap \mathrm{NsInt}(\Phi_\Lambda)$.

(viii) $\mathrm{NsInt}(\Psi_\Lambda) \cup \mathrm{NsInt}(\Phi_\Lambda) \subseteq \mathrm{NsInt}(\Psi_\Lambda \cup \Phi_\Lambda)$.

Definition 2.10 (Closure of a Neutrosophic Soft Set [21]). *Let* $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$ *be an NSTS and* $\Psi_\Lambda \in \mathrm{NSS}(\Omega_\Lambda)$. *The* closure *of* Ψ_Λ, *denoted by* $\mathrm{NsCl}(\Psi_\Lambda)$, *is defined as*

$$\mathrm{NsCl}(\Psi_\Lambda) = \bigcap\{\Xi_\Lambda \mid \Xi_\Lambda \textit{ is neutrosophic soft closed and } \Xi_\Lambda \supseteq \Psi_\Lambda\}.$$

In other words, the closure is the intersection of all neutrosophic soft closed sets that contain Ψ_Λ.

Theorem 2.2 (Properties of Neutrosophic Soft Closure [21]). *Let* $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$ *be an NSTS and* $\Psi_\Lambda, \Phi_\Lambda \in \mathrm{NSS}(\Omega_\Lambda)$. *Then:*

(i) $\Psi_\Lambda \subseteq \mathrm{NsCl}(\Psi_\Lambda)$, *and* $\mathrm{NsCl}(\Psi_\Lambda)$ *is the smallest NSCS containing* Ψ_Λ.

(ii) If $\Psi_\Lambda \subseteq \Phi_\Lambda$, *then* $\mathrm{NsCl}(\Psi_\Lambda) \subseteq \mathrm{NsCl}(\Phi_\Lambda)$.

(iii) $\mathrm{NsCl}(\Psi_\Lambda)$ *is neutrosophic soft closed:* $\mathrm{NsCl}(\Psi_\Lambda) \in \mathcal{T}^{c}_{\mathrm{NSS}}$.

(iv) Ψ_Λ *is neutrosophic soft closed iff* $\mathrm{NsCl}(\Psi_\Lambda) = \Psi_\Lambda$.

(v) $\mathrm{NsCl}(\mathrm{NsCl}(\Psi_\Lambda)) = \mathrm{NsCl}(\Psi_\Lambda)$.

(vi) $\mathrm{NsCl}(\emptyset_\Lambda) = \emptyset_\Lambda$ *and* $\mathrm{NsCl}(1_\Lambda) = 1_\Lambda$.

(vii) $\mathrm{NsCl}(\Psi_\Lambda \cup \Phi_\Lambda) = \mathrm{NsCl}(\Psi_\Lambda) \cup \mathrm{NsCl}(\Phi_\Lambda)$.

(viii) $\mathrm{NsCl}(\Psi_\Lambda \cap \Phi_\Lambda) \subseteq \mathrm{NsCl}(\Psi_\Lambda) \cap \mathrm{NsCl}(\Phi_\Lambda)$.

Theorem 2.3 (Complement Relations in Neutrosophic Soft Sets [21]). *Let* $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$ *be an NSTS and* $\Psi_\Lambda \in \mathrm{NSS}(\Omega_\Lambda)$. *Then, the complement operations satisfy*

$$\big(\mathrm{NsCl}(\Psi_\Lambda)\big)^c = \mathrm{NsInt}(\Psi_\Lambda^c), \quad \big(\mathrm{NsInt}(\Psi_\Lambda)\big)^c = \mathrm{NsCl}(\Psi_\Lambda^c).$$

Definition 2.11 (Neutrosophic Soft T_0 and T_1 Spaces [20]). *Let* $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$ *be an NSTS, and let*

$$\nu^{\lambda}_{(\alpha,\beta,\gamma)}, \quad \mu^{\lambda'}_{(\alpha',\beta',\gamma')}$$

be two distinct neutrosophic soft points in Ω_Λ. *Then:*

a) ***(Neutrosophic soft*** T_0***-space)*** $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$ *is called a* neutrosophic soft T_0-space *if there exist neutrosophic soft open sets* $\Psi_\Lambda, \Phi_\Lambda \in \mathcal{T}_{\mathrm{NSS}}$ *such that either*

$$\nu^{\lambda}_{(\alpha,\beta,\gamma)} \in \Psi_\Lambda \quad \textit{and} \quad \mu^{\lambda'}_{(\alpha',\beta',\gamma')} \notin \Psi_\Lambda,$$

or

$$\mu^{\lambda'}_{(\alpha',\beta',\gamma')} \in \Phi_\Lambda \quad \textit{and} \quad \nu^{\lambda}_{(\alpha,\beta,\gamma)} \notin \Phi_\Lambda.$$

b) ***(Neutrosophic soft*** T_1***-space)*** $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$ *is called a* neutrosophic soft T_1-space *if there exist neutrosophic soft open sets* $\Psi_\Lambda, \Phi_\Lambda \in \mathcal{T}_{\mathrm{NSS}}$ *such that*

$$\nu^{\lambda}_{(\alpha,\beta,\gamma)} \in \Psi_\Lambda, \quad \nu^{\lambda}_{(\alpha,\beta,\gamma)} \notin \Phi_\Lambda,$$

and

$$\mu^{\lambda'}_{(\alpha',\beta',\gamma')} \in \Phi_\Lambda, \quad \mu^{\lambda'}_{(\alpha',\beta',\gamma')} \notin \Psi_\Lambda.$$

3 Neutrosophic Soft Generalized Closed Sets

The notion of neutrosophic soft generalized closed sets extends the idea of generalized closed sets to NSTSs. In this section, we present their formal definition together with related theorems and illustrative examples.

Definition 3.1 (Neutrosophic Soft Generalized Closed Set [23]). *Let* $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$ *be an NSTS and* $\Psi_\Lambda \in \mathrm{NSS}(\Omega_\Lambda)$. *Then* Ψ_Λ *is called a* neutrosophic soft generalized closed set (NS-gCS) *if, for every neutrosophic soft open set* $\mathcal{O}_\Lambda$ *containing* Ψ_Λ, *the closure of* Ψ_Λ *is also contained in* $\mathcal{O}_\Lambda$, *i.e.,*

$$\Psi_\Lambda \subseteq \mathcal{O}_\Lambda \;\Rightarrow\; \mathrm{NsCl}(\Psi_\Lambda) \subseteq \mathcal{O}_\Lambda.$$

Equivalently,

$$\Psi_\Lambda \textit{ is an NS-gCS} \iff \mathrm{NsCl}(\Psi_\Lambda) \subseteq \mathcal{O}_\Lambda \textit{ for all NSOSs } \mathcal{O}_\Lambda \textit{ with } \Psi_\Lambda \subseteq \mathcal{O}_\Lambda.$$

Theorem 3.1. *[30] In any NSTS* $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$, *every NSCS is also an NS-gCS.*

Proof. Let Ψ_Λ be an NSCS and suppose $\Psi_\Lambda \subseteq \mathcal{O}_\Lambda$, where $\mathcal{O}_\Lambda$ is an NSOS. By Theorem 2.2 (iv), which states that $\mathrm{NsCl}(\Psi_\Lambda) = \Psi_\Lambda$, we have

$$\mathrm{NsCl}(\Psi_\Lambda) = \Psi_\Lambda \subseteq \mathcal{O}_\Lambda.$$

Therefore, Ψ_Λ satisfies the definition of an NS-gCS. □

Remark 3.1. *The converse does not necessarily hold; an NS-gCS need not be an NSCS.*

Example 3.1 (An NS-gCS that is not closed)**.** *Let $\Omega = \{\nu_1, \nu_2\}$ be the universal set and $\Lambda = \{\lambda_1, \lambda_2\}$ the set of parameters. Consider the family*

$$\mathcal{T}_{\mathrm{NSS}} = \{\emptyset_\Lambda,\ 1_\Lambda,\ \Phi_\Lambda,\ \Theta_\Lambda\}$$

defined on Ω_Λ, where Φ_Λ and Θ_Λ are given by

$$\Phi_\Lambda = \begin{Bmatrix} (\lambda_1, \{\langle \nu_1, 0.6, 0.2, 0.4\rangle,\ \langle \nu_2, 0.4, 0.3, 0.6\rangle\}), \\ (\lambda_2, \{\langle \nu_1, 0.3, 0.6, 0.2\rangle,\ \langle \nu_2, 0.2, 0.5, 0.4\rangle\}) \end{Bmatrix},$$

$$\Theta_\Lambda = \begin{Bmatrix} (\lambda_1, \{\langle \nu_1, 0.2, 0.7, 0.5\rangle,\ \langle \nu_2, 0.3, 0.5, 0.6\rangle\}), \\ (\lambda_2, \{\langle \nu_1, 0.1, 0.4, 0.4\rangle,\ \langle \nu_2, 0.4, 0.6, 0.3\rangle\}) \end{Bmatrix}.$$

Then $\mathcal{T}_{\mathrm{NSS}}$ defines a neutrosophic soft topology, and thus $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$ is an NSTS. Now, consider the neutrosophic soft set

$$\Psi_\Lambda = \begin{Bmatrix} (\lambda_1, \{\langle \nu_1, 0.4, 0.4, 0.6\rangle,\ \langle \nu_2, 0.3, 0.2, 0.5\rangle\}), \\ (\lambda_2, \{\langle \nu_1, 0.2, 0.5, 0.5\rangle,\ \langle \nu_2, 0.5, 0.3, 0.4\rangle\}) \end{Bmatrix}.$$

With respect to this topology, suppose that $\mathrm{NsCl}(\Psi_\Lambda) = \Theta_\Lambda^c$. Since $\Psi_\Lambda \subseteq 1_\Lambda$ and $\Theta_\Lambda^c \subseteq 1_\Lambda$, it follows that Ψ_Λ is an NS-gCS. However, because $\mathrm{NsCl}(\Psi_\Lambda) = \Theta_\Lambda^c \neq \Psi_\Lambda$, the set Ψ_Λ is not an NSCS.

Theorem 3.2. *[30] Let Ψ_Λ and Φ_Λ be two NS-gCSs in $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$. Then the union $\Psi_\Lambda \cup \Phi_\Lambda$ is an NS-gCS in Ω_Λ.*

Proof. Since Ψ_Λ and Φ_Λ are NS-gCSs, for any NSOS $\mathcal{O}_\Lambda$ with $\Psi_\Lambda \subseteq \mathcal{O}_\Lambda$ and $\Phi_\Lambda \subseteq \mathcal{O}_\Lambda$, we have

$$\mathrm{NsCl}(\Psi_\Lambda) \subseteq \mathcal{O}_\Lambda \quad \text{and} \quad \mathrm{NsCl}(\Phi_\Lambda) \subseteq \mathcal{O}_\Lambda.$$

Since $\Psi_\Lambda \subseteq \mathcal{O}_\Lambda$ and $\Phi_\Lambda \subseteq \mathcal{O}_\Lambda$, it follows that

$$\Psi_\Lambda \cup \Phi_\Lambda \subseteq \mathcal{O}_\Lambda.$$

By Theorem 2.2 (vii),

$$\mathrm{NsCl}(\Psi_\Lambda \cup \Phi_\Lambda) \subseteq \mathrm{NsCl}(\Psi_\Lambda) \cup \mathrm{NsCl}(\Phi_\Lambda) \subseteq \mathcal{O}_\Lambda,$$

which shows that $\Psi_\Lambda \cup \Phi_\Lambda$ is an NS-gCS. □

Theorem 3.3. *[30] Let Ψ_Λ and Φ_Λ be two NS-gCSs in $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$. Then*

$$\mathrm{NsCl}(\Psi_\Lambda \cap \Phi_\Lambda) \subseteq \mathrm{NsCl}(\Psi_\Lambda) \cap \mathrm{NsCl}(\Phi_\Lambda).$$

Proof. Since Ψ_Λ and Φ_Λ are NS-gCSs, for any NSOS $\mathcal{O}_\Lambda$ with $\Psi_\Lambda \subseteq \mathcal{O}_\Lambda$ and $\Phi_\Lambda \subseteq \mathcal{O}_\Lambda$, we have

$$\mathrm{NsCl}(\Psi_\Lambda) \subseteq \mathcal{O}_\Lambda \quad \text{and} \quad \mathrm{NsCl}(\Phi_\Lambda) \subseteq \mathcal{O}_\Lambda.$$

Since $\Psi_\Lambda \cap \Phi_\Lambda \subseteq \Psi_\Lambda$ and $\Psi_\Lambda \cap \Phi_\Lambda \subseteq \Phi_\Lambda$, by Theorem 2.2 (viii) we obtain

$$\mathrm{NsCl}(\Psi_\Lambda \cap \Phi_\Lambda) \subseteq \mathrm{NsCl}(\Psi_\Lambda) \cap \mathrm{NsCl}(\Phi_\Lambda),$$

which completes the proof. □

Remark 3.2. *The intersection of two NS-gCSs need not be an NS-gCS.*

Example 3.2. *Let $\Omega = \{\omega_1, \omega_2\}$ be the universe and $\Lambda = \{\lambda_1, \lambda_2\}$ the parameter set. Consider the neutrosophic soft topology*

$$\mathcal{T}_{\mathrm{NSS}} = \{\emptyset_\Lambda,\ 1_\Lambda,\ H_\Lambda\}$$

on Ω_Λ, where

$$H_\Lambda = \begin{Bmatrix} (\lambda_1, \{\langle \omega_1, 0.4, 0.6, 0.5\rangle,\ \langle \omega_2, 0.3, 0.7, 0.4\rangle\}), \\ (\lambda_2, \{\langle \omega_1, 0.5, 0.5, 0.3\rangle,\ \langle \omega_2, 0.2, 0.6, 0.2\rangle\}) \end{Bmatrix}.$$

Define the neutrosophic soft sets A_Λ and B_Λ in $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$ by

$$A_\Lambda = \begin{Bmatrix} (\lambda_1, \{\langle \omega_1, 0.7, 0.5, 0.4\rangle,\ \langle \omega_2, 0.8, 0.4, 0.3\rangle\}), \\ (\lambda_2, \{\langle \omega_1, 0.6, 0.6, 0.2\rangle,\ \langle \omega_2, 0.4, 0.5, 0.3\rangle\}) \end{Bmatrix},$$

$$B_\Lambda = \begin{Bmatrix} (\lambda_1, \{\langle \omega_1, 0.6, 0.8, 0.6\rangle,\ \langle \omega_2, 0.7, 0.6, 0.5\rangle\}), \\ (\lambda_2, \{\langle \omega_1, 0.8, 0.4, 0.3\rangle,\ \langle \omega_2, 0.3, 0.7, 0.4\rangle\}) \end{Bmatrix}.$$

Their intersection is

$$A_\Lambda \cap B_\Lambda = \begin{Bmatrix} (\lambda_1, \{\langle \omega_1, 0.6, 0.5, 0.6\rangle,\ \langle \omega_2, 0.7, 0.4, 0.5\rangle\}), \\ (\lambda_2, \{\langle \omega_1, 0.6, 0.4, 0.2\rangle,\ \langle \omega_2, 0.3, 0.6, 0.4\rangle\}) \end{Bmatrix}.$$

By Definition 3.1, one checks that both A_Λ and B_Λ are NS-gCSs in Ω_Λ (their respective neutrosophic soft closures lie inside 1_Λ and satisfy the required conditions). However, for the intersection we obtain

$$\mathrm{NsCl}(A_\Lambda \cap B_\Lambda) = 1_\Lambda,$$

and since $1_\Lambda \not\subseteq H_\Lambda$, it follows that $A_\Lambda \cap B_\Lambda$ is not *an NS-gCS in Ω_Λ.*

Theorem 3.4. *Let $\Psi_\Lambda \subseteq \Phi_\Lambda \subseteq \Omega_\Lambda$ in $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$. If Ψ_Λ is an NS-gCS in Φ_Λ, and Φ_Λ is an NS-gCS in Ω_Λ, then Ψ_Λ is also an NS-gCS in Ω_Λ.*

Proof. **Step 1: NS-gCS property in Φ_Λ** Since Ψ_Λ is an NS-gCS in Φ_Λ, we have

$$\Psi_\Lambda = \mathrm{NsCl}_{\Phi_\Lambda}(\Psi_\Lambda) \cap \Phi_\Lambda,$$

where $\mathrm{NsCl}_{\Phi_\Lambda}(\Psi_\Lambda)$ denotes the NS-closure of Ψ_Λ in Φ_Λ.

Step 2: NS-gCS property in Ω_Λ Since Φ_Λ is an NS-gCS in Ω_Λ,

$$\Phi_\Lambda = \mathrm{NsCl}_{\Omega_\Lambda}(\Phi_\Lambda).$$

Step 3: Monotonicity of NS-closure By monotonicity of NsCl, we get

$$\mathrm{NsCl}_{\Omega_\Lambda}(\Psi_\Lambda) \subseteq \mathrm{NsCl}_{\Omega_\Lambda}(\Phi_\Lambda) = \Phi_\Lambda.$$

Step 4: Concluding NS-gCS in Ω_Λ Hence,

$$\mathrm{NsCl}_{\Omega_\Lambda}(\Psi_\Lambda) \cap \Omega_\Lambda = \Psi_\Lambda,$$

showing that Ψ_Λ is an NS-gCS in Ω_Λ. □

Theorem 3.5. *In the NSTS $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$, if Ψ_Λ is an NS-gCS and Θ_Λ is an NSCS in Ω_Λ, then the intersection $\Psi_\Lambda \cap \Theta_\Lambda$ is an NS-gCS in Ω_Λ.*

Proof. Since Ψ_Λ is an NS-gCS in Ω_Λ, we have $\Psi_\Lambda = \mathrm{NsCl}(\Psi_\Lambda) \cap \Omega_\Lambda$, and since Θ_Λ is an NSCS in Ω_Λ, its complement Θ_Λ^c is an NS-gOS. By monotonicity of NsCl,

$$\mathrm{NsCl}(\Psi_\Lambda \cap \Theta_\Lambda) \subseteq \mathrm{NsCl}(\Psi_\Lambda) \cap \mathrm{NsCl}(\Theta_\Lambda) = \Psi_\Lambda \cap \Theta_\Lambda.$$

Hence,

$$\mathrm{NsCl}(\Psi_\Lambda \cap \Theta_\Lambda) \cap \Omega_\Lambda = \Psi_\Lambda \cap \Theta_\Lambda,$$

showing that $\Psi_\Lambda \cap \Theta_\Lambda$ is an NS-gCS in Ω_Λ. □

Theorem 3.6. *[30] Let Ψ_Λ be an NS-gCS in $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$, and let $\Psi_\Lambda \subseteq \Phi_\Lambda \subseteq \mathrm{NsCl}(\Psi_\Lambda)$. Then Φ_Λ is an NS-gCS in Ω_Λ.*

Theorem 3.7. *Let $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$ be an NSTS over Ω_Λ, and let $\Psi_\Lambda \subseteq \Omega_\Lambda$. Then Ψ_Λ is an NS-gCS if and only if $\mathrm{NsCl}(\Psi_\Lambda) - \Psi_\Lambda$ contains no non-empty NSCS.*

Proof. Suppose there exists a neutrosophic soft closed subset $\Theta_\Lambda \subseteq \mathrm{NsCl}(\Psi_\Lambda) - \Psi_\Lambda$. Then

$$\Psi_\Lambda \subseteq \Theta_\Lambda^c.$$

Since Ψ_Λ is an NS-gCS, we have

$$\mathrm{NsCl}(\Psi_\Lambda) \subseteq \Theta_\Lambda^c \quad \Longrightarrow \quad \Theta_\Lambda \subseteq \mathrm{NsCl}(\Psi_\Lambda) \cap (\mathrm{NsCl}(\Psi_\Lambda))^c = \emptyset_\Lambda,$$

which implies that Θ_Λ is empty.

Conversely, suppose $\Psi_\Lambda \subseteq \mathcal{O}_\Lambda$, where $\mathcal{O}_\Lambda$ is an NSOS, but $\mathrm{NsCl}(\Psi_\Lambda) \not\subseteq \mathcal{O}_\Lambda$. Then

$$\mathrm{NsCl}(\Psi_\Lambda) \cap \mathcal{O}_\Lambda^c$$

would be a non-empty NSCS contained in $\mathrm{NsCl}(\Psi_\Lambda) - \Psi_\Lambda$, contradicting the assumption. Hence the result follows. □

Definition 3.2. *[30] Let $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$ be an NSTS over Ω_Λ and $\Phi_\Lambda \in \mathrm{NSS}(\Omega_\Lambda)$. If the complement Φ_Λ^c is an NS-gCS in Ω_Λ, then Φ_Λ is called a* neutrosophic soft generalized open set (NS-gOS).

Theorem 3.8. *Let* $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$ *be an NSTS over* Ω_Λ *and* $\Phi_\Lambda \in \mathrm{NSS}(\Omega_\Lambda)$*. Then* Φ_Λ *is an NS-gOS if and only if every neutrosophic soft closed set* $\Theta_\Lambda \subseteq \Phi_\Lambda$ *is contained in* $\mathrm{NsInt}(\Phi_\Lambda)$*.*

Proof. If Φ_Λ is an NS-gOS, i.e., Φ_Λ^c is an NS-gCS, then by the properties of NS-gCS and NS-interior, any neutrosophic soft closed set $\Theta_\Lambda \subseteq \Phi_\Lambda$ must satisfy $\Theta_\Lambda \subseteq \mathrm{NsInt}(\Phi_\Lambda)$. Conversely, if every neutrosophic soft closed set contained in Φ_Λ lies in $\mathrm{NsInt}(\Phi_\Lambda)$, then Φ_Λ^c fulfills the NS-gCS property, showing that Φ_Λ is an NS-gOS. □

Example 3.3. *Let* $\Omega = \{\omega_1, \omega_2\}$ *be the universe and* $\Lambda = \{\lambda_1, \lambda_2\}$ *the parameter set. Consider the NSTS* $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$ *as in Example 3.2 (with the same underlying topology). Define the neutrosophic soft set*

$$\Psi_\Lambda = \begin{Bmatrix} (\lambda_1, \{\langle \omega_1, 0.1, 0.3, 0.6\rangle,\ \langle \omega_2, 0.2, 0.4, 0.8\rangle\}), \\ (\lambda_2, \{\langle \omega_1, 0.2, 0.2, 0.7\rangle,\ \langle \omega_2, 0.3, 0.3, 0.5\rangle\}) \end{Bmatrix}.$$

By Definition 3.2, since $\emptyset_\Lambda \subseteq \Psi_\Lambda$ *and*

$$\mathrm{NsInt}(\Psi_\Lambda) = \emptyset_\Lambda,$$

we conclude that Ψ_Λ *is an* NS-gOS *in* Ω_Λ*.*

Theorem 3.9. *[30] Let* $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$ *be an NSTS over* Ω_Λ*, and let* $\Psi_\Lambda, \Phi_\Lambda \subseteq \Omega_\Lambda$*. If* Ψ_Λ *and* Φ_Λ *are discrete NS-gOS, then the intersection* $\Psi_\Lambda \cap \Phi_\Lambda$ *is also an NS-gOS in* Ω_Λ*.*

Remark 3.3. *The union of two neutrosophic soft generalized open sets is not necessarily an NS-gOS, as demonstrated below.*

Example 3.4. *Let* $\Omega = \{\omega_1, \omega_2\}$ *and* $\Lambda = \{\lambda_1, \lambda_2\}$*. Consider the neutrosophic soft topological space* $(\Omega, \mathcal{T}_{\mathrm{NSS}}, \Lambda)$ *as described in Example 3.2. Define*

$$G_\Lambda = \begin{Bmatrix} (\lambda_1, \{\langle \omega_1, 0.3, 0.4, 0.6\rangle,\ \langle \omega_2, 0.2, 0.5, 0.7\rangle\}), \\ (\lambda_2, \{\langle \omega_1, 0.4, 0.3, 0.5\rangle,\ \langle \omega_2, 0.3, 0.4, 0.6\rangle\}) \end{Bmatrix},$$

$$K_\Lambda = \begin{Bmatrix} (\lambda_1, \{\langle \omega_1, 0.5, 0.2, 0.4\rangle,\ \langle \omega_2, 0.4, 0.3, 0.8\rangle\}), \\ (\lambda_2, \{\langle \omega_1, 0.2, 0.6, 0.5\rangle,\ \langle \omega_2, 0.1, 0.5, 0.7\rangle\}) \end{Bmatrix}.$$

Then their union is

$$G_\Lambda \cup K_\Lambda = \begin{Bmatrix} (\lambda_1, \{\langle \omega_1, 0.5, 0.4, 0.4\rangle,\ \langle \omega_2, 0.4, 0.5, 0.7\rangle\}), \\ (\lambda_2, \{\langle \omega_1, 0.4, 0.6, 0.5\rangle,\ \langle \omega_2, 0.3, 0.5, 0.7\rangle\}) \end{Bmatrix}.$$

Although both G_Λ *and* K_Λ *are NS-gOS in* Ω_Λ*, their union fails to be an NS-gOS, since*

$$(G_\Lambda \cup K_\Lambda)^c \not\subseteq \mathrm{NsInt}(G_\Lambda \cup K_\Lambda) = \emptyset_\Lambda.$$

Corollary 3.1. *[30] Let* Ψ_Λ *and* Φ_Λ *be discrete NS-gCSs in* Ψ_Λ*. If* Ψ_Λ^c *and* Φ_Λ^c *are discrete, then* $\Psi_\Lambda \cap \Phi_\Lambda$ *is an NS-gCS, because* $(\Psi_\Lambda \cap \Phi_\Lambda)^c = \Psi_\Lambda^c \cup \Phi_\Lambda^c$ *is an NS-gOS, and Theorem 3.9 applies.*

Theorem 3.10. *Let* $(\Omega, \mathcal{T}_{\text{NSS}}, \Lambda)$ *be a neutrosophic soft topological space. Suppose* Ψ_Λ *is an NS-gOS in* Ω_Λ*, and let* $\Phi_\Lambda \subseteq \Omega_\Lambda$ *satisfy*

$$\Psi_\Lambda^c \subseteq \Phi_\Lambda^c \subseteq \text{NsCl}(\Psi_\Lambda^c).$$

Then Φ_Λ *is also an NS-gOS in* Ω_Λ*.*

Proof. Consider the complements of the sets. Since Ψ_Λ is an NS-gOS, its complement Ψ_Λ^c is an NS-gCS. From the assumption, Φ_Λ^c lies between Ψ_Λ^c and its neutrosophic soft closure:

$$\Psi_\Lambda^c \subseteq \Phi_\Lambda^c \subseteq \text{NsCl}(\Psi_\Lambda^c).$$

By Theorem 3.6, any neutrosophic soft set situated between an NS-gCS and its closure is itself an NS-gCS. Consequently, Φ_Λ^c is an NS-gCS, and taking the complement, we deduce that Φ_Λ is an NS-gOS in Ω_Λ. □

Theorem 3.11. *Let* $\Psi_\Lambda \in \text{NSS}(\Omega_\Lambda)$*. Then* Ψ_Λ *is an NS-gCS if and only if its complement* Ψ_Λ^c *is an NS-gOS in* Ω_Λ *relative to* $\text{NsCl}(\Psi_\Lambda)$*.*

Proof. ($\Rightarrow$) Suppose Ψ_Λ is an NS-gCS. By definition, $\Psi_\Lambda^c = \text{NsCl}(\Psi_\Lambda) - \Psi_\Lambda$ (see Definition 3.1). Let $U_\Lambda \subseteq \Psi_\Lambda^c$ be any NSCS. Then, according to the definition of NS-gCS and Theorem 3.3, U_Λ must be the empty set $\emptyset_\Lambda$. Hence, by Definition 3.2, Ψ_Λ^c is an NS-gOS in $\text{NsCl}(\Psi_\Lambda)$.

($\Leftarrow$) Conversely, assume that Ψ_Λ^c is an NS-gOS in $\text{NsCl}(\Psi_\Lambda)$. Let O_Λ be any NSOS such that $\Psi_\Lambda \subseteq O_\Lambda$. Then

$$\text{NsCl}(\Psi_\Lambda) \cap O_\Lambda^c \subseteq \Psi_\Lambda^c$$

is an NSCS. By the NS-gOS property of Ψ_Λ^c (Definition 3.2), we have

$$\text{NsCl}(\Psi_\Lambda) \cap O_\Lambda^c \subseteq \text{NsInt}(\Psi_\Lambda^c) = \emptyset_\Lambda.$$

Therefore, $\text{NsCl}(\Psi_\Lambda) \subseteq O_\Lambda$, which shows that Ψ_Λ is an NS-gCS. □

In this part, we present some applications of neutrosophic soft generalized closed sets (NS-gCSs) in the context of separation axioms in NSTSs. We focus on the $T_{1/2}$ separation property and its implications.

Definition 3.3. *an NSTS* $(\Omega, \mathcal{T}_{\text{NSS}}, \Lambda)$ *is called a* neutrosophic soft-$T_{1/2}$ space *(in short,* N-$({}^sT_{1/2})$*) if every NS-gCS in* $(\Omega, \mathcal{T}_{\text{NSS}}, \Lambda)$ *is also a neutrosophic soft closed set (NSCS).*

Example 3.5. *Let* $\Omega = \{\nu_1, \nu_2\}$ *and* $\Lambda = \{\lambda_1, \lambda_2\}$*. Consider the NSTS*

$$(\Omega, \mathcal{T}_{\text{NSS}}, \Lambda), \qquad \mathcal{T}_{\text{NSS}} = \{\emptyset_\Lambda, 1_\Lambda, \Phi_\Lambda\},$$

where

$$\Phi_\Lambda = \{(\lambda_1, \{\langle \nu_1, 0.8, 0.8, 0.8\rangle\})\}.$$

Then $(\Omega, \mathcal{T}_{\text{NSS}}, \Lambda)$ *is an* N-$({}^sT_{1/2})$ *space, since the only NS-gCS* Φ_Λ *is also an NSCS.*

Theorem 3.12. *If an NSTS* $(\Omega, \mathcal{T}_{\text{NSS}}, \Lambda)$ *is* N-$({}^sT_1)$*, then it is* N-$({}^sT_{1/2})$*.*

Proof. In an N-($^sT_1$) space, every singleton neutrosophic soft set is an NSCS. Therefore, every NS-gCS is automatically an NSCS, and the space is N-($^sT_{1/2}$). □

Example 3.6. *Let $\Omega = \{\nu_1, \nu_2\}$ and $\Lambda = \{\lambda_1, \lambda_2\}$, and consider the neutrosophic soft topology*

$$\mathcal{T}_{\text{NSS}} = \{\emptyset_\Lambda, 1_\Lambda, \Phi_\Lambda\},$$

where

$$\Phi_\Lambda = \{(\lambda_1, \{\langle \nu_1, 0.8, 0.8, 0.8\rangle, \langle \nu_2, 0.7, 0.7, 0.7\rangle\})\}.$$

Then $(\Omega, \mathcal{T}_{\text{NSS}}, \Lambda)$ is N-($^sT_1$) because each singleton neutrosophic soft point can be separated from every NSCS. Consequently, every NS-gCS is an NSCS, so the space is also N-($^sT_{1/2}$).

Theorem 3.13. *If an NSTS is N-($^sT_{1/2}$), then it is N-($^sT_0$).*

Proof. Since every NS-gCS is an NSCS in an N-($^sT_{1/2}$) space, for any two distinct neutrosophic soft points, there exists a neutrosophic soft open set containing one but not the other, which satisfies the T_0 separation condition. □

4 Conclusion

In this paper, the concept of NS-gCSs has been introduced, and their fundamental properties have been investigated in detail. Key theorems within NSTSs have been established, and these notions have been illustrated through concrete examples, thereby providing a solid theoretical foundation for NS-gCSs. The present study is expected to serve as a basis for further research on generalized neutrosophic soft structures. In the future, the generalization of alpha, semi, pre, and beta sets within neutrosophic soft topological spaces could be explored. The proposed structures are thought to enable a systematic study of topological properties, including continuity, compactness, connectedness, and separation axioms, within the neutrosophic soft context. Although the present work is primarily theoretical, the developed framework is considered to hold significant potential for advancing neutrosophic soft topology, as well as for applications in decision-making, algebraic structures, and other applied sciences. The concepts and results presented in this study are expected to provide a valuable starting point for such further investigations.

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