# A study of the time evolution of cosmological parameters in the framework of an anisotropic Kaluza-Klein metric, using a negative power of linear function as a scaling factor.

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**Abstract**: The current study seeks to estimate the temporal dependence of some cosmological parameters in flat space using an anisotropic Kaluza-Klein metric. The field equations for this study were derived from the metric by assuming a power-law relationship between the normal scale factor and the scale factor for the extra dimension. We employed a new empirical scale factor, $a = \frac{B}{(1+\alpha t)^{\beta}}$ to generate time-dependent formulations for cosmological parameters. The reason for selecting this scale factor is that it generates an expression for the deceleration parameter, which changes sign over time from positive to negative, indicating a transition of the universe from an initial state of decelerated expansion to one of accelerated expansion, as inferred from astrophysical observations. We have graphically illustrated the evolution of some cosmological parameters in terms of relative time, denoted as $\frac{t}{t_0}$, where $t_0$ is the universe's current age. The work reveals that the dynamical cosmological constant (Λ) is initially negative and gradually decreases over time. The fifth dimension's pressure-density relationship has been represented using a skewness parameter(δ) , which decreases over time. The anisotropy factor was determined in this study, and its numerical value was shown to be decreasing with time, indicating that the cosmos is moving toward periods of steadily decreasing anisotropy.

**Keywords:** scale factor; Kaluza-Klein theory; Cosmological parameter (Λ); Dark energy; Anisotropy; empirical function.

## 1. INTRODUCTION:

The study of the time development of cosmological parameters using an anisotropic Kaluza-Klein metric is an attractive area of research that integrates principles from cosmology and higher-dimensional gravity theories. The Kaluza-Klein hypothesis, which was originally developed to reconcile gravity and electromagnetism, proposes the presence of additional spatial dimensions beyond the usual three. This paradigm allows for a more complex structure of space-time, which could explain a variety of cosmic events. In cosmology, anisotropic metrics are vital for describing the universe's dynamics, particularly in conditions when homogeneity and isotropy (conventional cosmological model assumptions) are insufficient, such as in the early beginning or near cosmological singularities. Researchers hope to learn more about the nature of dark energy, the expansion of the universe, and possible pathways for structure development by examining the evolution of parameters such as the scale factor, curvature, and energy density over time under an anisotropic Kaluza-Klein metric.Usually, this study entails obtaining field equations from the higher-dimensional Einstein-Hilbert action, examining how they may affect the evolution of cosmic parameters, and determining how anisotropies may affect the universe's general dynamics.

These investigations may also take into account different matter contents, such as fluid distributions or scalar fields, which could contribute to the anisotropic effects seen.All things considered, using this framework to analyze the time history of cosmological parameters improves our comprehension of the evolution of the universe and the basic properties of spacetime, which may result in new predictions and discoveries in both theoretical and observational cosmology.

Based on the observational results from supernova 1a, it was determined that the current period of accelerated expansion of the universe is caused by a negative pressure created by an exotic kind of energy known as dark energy (DE) [1, 2]. Only thorough examinations will be able to determine how this enigmatic DE operates. According to a detailed examination of supernova data, the universe has transitioned from a phase of decelerated expansion to accelerated expansion, which has caused the deceleration parameter's sign to change from positive to negative [3-5]. Approaches to determining the nature of cosmic acceleration are often found in the extensive scientific literature in two primary methods. Using modified theories of gravity which are based on variations of Einstein's theory of general relativity, to build mathematical models and investigate their properties is one method. The alternative is to use dark energy models to study the cosmic observations. In several cosmological models, a quantity known as the cosmological constant (represented by $\Lambda$) has been claimed to reflect DE.

In the scientific literature, there are several models of dark energy, including quintom, k-essence, phantom, and quintessence [6–9]. Despite being included as a constant parameter in Einstein's theory [10], $\Lambda$ is currently considered a time-dependent quantity [11] due to certain restrictions related to the coincidence problem and the cosmological constant problem. Researchers have developed theories [12–14] and scalar tensor models such as the Saez-Ballester (SB) and Brans-Dicke (BD) theories of gravity [15,16] by altering Einstein's theory of gravity in different ways. By building models of cosmic phenomena utilising DE, a wide range of investigations have been conducted [17–20].

Two scientists, Kaluza and Klein, put forth a novel hypothesis in the first half of the previous century to combine electromagnetic and gravitational forces, and it has been known as Kaluza-Klein (KK) theory ever since [21, 22]. The fifth dimension, which served as a bridge between the two forces, was discussed in this theory. Chodos and Detweiler demonstrated a shrinkage of this new dimension with time using a five-dimensional model based on KK theory [23]. It is theoretically shown that there was a period of a multidimensional state that preceded the present four-dimensional representation of the universe. With the universe's evolution, the additional dimension gets smaller, and current scientific methods are unable to detect it.

These events have prompted numerous scholars to conduct studies by developing models with higher dimensions. The KK theory can be viewed as a five-dimensional version of the general theory of relativity. Chodos & Detweller [23], Witten [24], Appelquist et al. [25], Appelquist & Chodos [26], and Marchiano [27] conducted theoretical investigations that are regarded as highly significant in this subject. The current work was motivated by N. I. Jain's description of an anisotropic dark energy model based on KK theory [28]. J.A. Ferrari's research of a spherically symmetric charged system yielded an approximate solution to Kaluza-Klein's equations [29].

This investigation was conducted to determine how a test particle behaves in a field of force created by a charged particle. According to this work, Kaluza-Klein's theory permits us to calculate Lorentz force adjustments. These experimental data verify the five-dimensional relativity of the KK framework. Kalligas et al. [30] found that the existence of an extra spacetime dimension can be experimentally verified. This investigation produced a set of equations that includes terms related to the existence of an extra dimension.

Using data from solar system measurements, it was determined that the terms indicating the fifth dimension are exceedingly minor in relation to the ordinary dimensions of spacetime in our regio n of space.It has been discovered that the parameters related to the KK theory cannot be viewed a s universal constants, and that they can vary from place to place based on the local properties of matter.Dzhunushaliev et al. discovered many Kaluza- Klein space time solutions that were not asymptotically flat and contained both electric and Magnetic charges [31]. It was demonstrated that these solutions might be seen as virtual quantum handles (wormholes) in mode ls of space-time foam.It has been proven that, in the presence of a sufficiently massive external magnetic and/or electric field, these solutions may be inflated from a quantum to a classical state. This study suggests that with a multidimensional gravity, there may be an experimental indicator for higher dimensions. Jean Paul Mbelek's new work [32] suggests that integrating an external scalar field ( $\psi$) can improve the theory-based five-dimensional Kaluza-Klein metric. This approach resulted in observational results (for laboratory experiments as well as astrophysics and cosmology) that are congruent with theoretical discoveries.

According to the study, a torque acting on a torsion pendulum was detected, which is consistent with theoretical expectations. Based on a recent experimental research by Tajmar and Williams [33], a macroscopic interpretation of the fifth dimension of the Kaluza-Klein theory was produced. This experiment was conducted to validate a key feature of theoretical discoveries that reveal the fifth dimension to be related to electric charge. They used Kaluza-Klein theory to analyze their observations of the time dilation phenomenon in an electrically charged clock.

They used Kaluza-Klein theory to understand their observations of the time dilation phenomenon in an electrically charged clock. They emphasized that the five-dimensional metric should have a time-like signature in order to provide a classical interpretation for the extra dimension. The current work aims to analyze the nature of temporal dependency in various cosmological parameters using an anisotropic Kaluza-Klein spacetime. The research includes a time-dependent cosmological term ($\Lambda$). The Kaluza-Klein metric, which is being used here, assumes that there is a power-law relationship ($a = A^n$) between the normal scale factor ($a$) and the scale factor denoting the additional dimension ($A$). To solve the field equations, we employed a scale factor ansatz $\left(a = \frac{B}{(1+\alpha t)^{\beta}}\right)$.The reason for opting for this function is that the deceleration parameter $\left(q = -\frac{\ddot{a}a}{\dot{a}^2}\right)$ derived from this scale factor undergoes a signature flip with time from positive to negative, which is consistent with the fact that the universe's current phase of accelerated expansion was preceded by a phase of decelerated expansion [3-5]. Using this empirical scale factor, we have derived expressions for some cosmological quantities such as the parameter of Hubble ($H$), deceleration parameter ($q$), energy density ($\rho$), cosmological constant ($\Lambda$), equation of state (EoS) parameter ($\omega$), skewness parameter ($\delta$), and anisotropy factor ($\frac{\sigma^2}{\theta}$).

We demonstrated their time variation by graphing them visually as functions of the relative cosmic time ($\frac{t}{t_0}$), where $t_0$ indicates the age of the universe at the current time.

Our analysis of the skewness parameter ($\delta$) and anisotropy factor ($\frac{\sigma^2}{\theta}$) indicates a trend towards lower anisotropy levels. In our analysis, the dynamical cosmological term ($\Lambda$) is negative (growing less negative with time) and varies slowly in the present-day universe, indicating a sluggish rise in the dark energy content, which has been suggested to cause cosmic acceleration. This article includes five sections, including the introduction. Sections 2 discuss the field equations and their solutions. Section three deals with the determination of cosmological quantities. Sections 4 and 5 discuss the theoretical investigation's results and conclusions.

## 2. FIELD EQUATIONS AND THE METRIC.

In order to obtain the cosmological field equations, we have used the Kaluza-Klein space-time of the following type [34].

$$ds^2 = -ds^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 sin^2\theta\, d\phi^2\right] + A^2(t) d\psi^2 \quad (1)$$

$a(t)$ and $A(t)$ are the 4th and 5th dimension scale factors, respectively. The symbol $k$ is a measure of the spatial curvature, having the values $-1$ , $0$ , and $1$ respectively for the open, flat and closed universes. The energy-momentum tensor $\left(T_j^i\right)$, for the anisotropic space-time metric represented by equation (1), is given below [35].

$$T_j^i = diag(T_0^0, T_1^1, T_2^2, T_3^3, T_4^4\,) = diag\left(-\rho, p, p, p, p_\psi\right) \quad (2)$$

In equation (2), the symbols $\rho$ and $p$ represent the energy density and pressure of the cosmic fluid (dark energy) that penetrates the universe. The symbol $p_\psi$ represents the pressure related to the additional dimension. The barotropic equation of state (EoS) parameter for normal dimensions is $\omega = \frac{p}{\rho}$. In Kaluza-Klein theory, we used the equation $p_\psi = (\delta + \omega)\rho$as the directional equation of state for the fifth dimension, where $\delta$ is the skewness parameter that represents the deviation from the normal equation-of-state parameter $\omega$ [28, 36-41]. The parameter $\delta$ measures the divergence from isotropy. Thus, the energy-momentum tensor of equation (2) can be represented as:

$$T_j^i = diag(-\rho, \omega\rho, \omega\rho, \omega\rho, (\omega + \delta)\rho) \quad (3)$$

This study examines the relationship between $\omega$ and $\delta$ throughout time. Gravitational field equations are derived from the following equation.

$$G_j^i = R_j^i - \frac{1}{2} R\delta_j^i = -8\pi G T_j^i + \Lambda\delta_j^i \quad (4)$$

To create the field equations, we employed an ansatz for the fifth-dimension scale factor ($A$), which is $A = a^n$ [42]. We also utilized $8\pi G = c = 1$ and $k = 0$ (i.e., flat space). Combining equations (1), (3), and (4) yields the following field equations.

$$(n+2)\dot{H} + (n^2 + 2n + 3)H^2 = -\omega\rho + \Lambda \tag{5}$$

$$3\dot{H} + 6H^2 = -(\omega + \delta)\rho + \Lambda \tag{6}$$

$$3(n+1)H^2 = \rho + \Lambda \tag{7}$$

Divergence of Einstein's tensor can be expressed as,

$$\left(R_j^i - \frac{1}{2}R\delta_j^i\right)_{;j} = \left(-T_j^i + \Lambda\delta_j^i\right)_{;j} = 0 \tag{8}$$

Using equation (8), the equation describing energy conservation [35] is given by,

$$\dot{\rho} + 3(\rho + p)H + n(\rho + p_\psi)H + \dot{\Lambda} = 0 \tag{9}$$

Replace $p = \omega\rho$ and $p_\psi = (\omega + \delta)\rho$ in (9)

We get

$$\dot{\rho} + (3+n)H + (1+\omega)\rho H + n\rho\delta H + \dot{\Lambda} = 0 \tag{10}$$

Equation (10) can be represented as the sum of two equations, (11) and (12), as shown below.

$$\dot{\rho} + (3+n)(1+\omega)\rho H = Q \tag{11}$$

$$n\rho\delta H + \dot{\Lambda} = -Q \tag{12}$$

In equations (11) and (12), $Q$ is an arbitrary parameter. Subtracting equation (6) from equation (5), we get,

$$(n-1)\dot{H} + (n^2 + 2n - 3)H^2 = \rho\delta \tag{13}$$

Substitution for $\rho\delta$ in equation (12), based on equation (13), leads to the following differential equation.

$$\dot{\Lambda} = -Q - nH(n-1)\left[H + (n+3)H^2\right] \tag{14}$$

## 3. SOLUTION OF THE FIELD EQUATIONS USING NEGATIVE POWER OF LINEAR FUNCTION AS EMPIRICAL SCALE FACTOR

To solve the field equations, we have used the

$$a = \frac{B}{(1+\alpha t)^\beta} \tag{15}$$

Where the constants $B$ , $\alpha$ , $\beta > 0$

Using this scale factor (eqn. 15) results in a deceleration parameter (equation 17) that changes sign over time from positive to negative. This aligns with recent astrophysical observations [ref. nos. 3-5] that show a transition from decelerated to accelerated expansion of the universe. A hybrid scale factor, combining an exponential and power-law function of time, is commonly used for similar purposes. It has been employed in various recent cosmological research [43–49]. Some studies [50-55] have used hyperbolic functions of time as empirical scale factors. possessing the same property (i.e., the shift from slowdown to acceleration) as cosmic expansion. The parameter $B$ in the scale factor expression (eqn. 15) does not occur in the equations describing the Hubble parameter and deceleration parameter (eqns. 16 & 17), which are expressed as $H$ and $q$, respectively. The functions of $H$ and $q$ are independent of $H$. Equations (5), (6), and (7) use the Hubble parameter ($H$) and its time derivative, making them independent of the parameter $B$. The parameter $B$ is not used in any of the expressions in this article, save for the scaling factor (eqn. 15). The Hubble parameter ($H$) can be calculated using our empirical scaling factor (eqn. 15) as follows:

$$\dot{a} = \frac{-\alpha\beta B}{(1+\alpha t)^{\beta+1}}$$

$$H = \frac{\dot{a}}{a} = \frac{-\alpha\beta}{(1+\alpha t)} \tag{16}$$

Based on our empirical scale factor (eqn. 15), the deceleration parameter ($q$) is given by,

$$\dot{a} = -\alpha\beta(1+\alpha t)^{-1}a$$

$$\ddot{a} = -\alpha\beta[-(1+\alpha t)^{-2}a\alpha + (1+\alpha t)^{-1}\dot{a}]$$

$$q = \frac{\ddot{a}\,a}{\dot{a}^2} = \left(\frac{1}{\beta}+1\right) \tag{17}$$

In the present article, we have used the symbols $H$ and $q$, which stand for the values of $H$ and $q$, respectively at the present time (i.e., $t = t_0$ ) where $t$ denotes the age of the universe ( $t = 13.7 \times 10^9$ years). Putting $H = H_0,\ q = q_0$ and., $t = t_0$ in equations (16) and (17), we get

$$H_0 = \frac{-\alpha\beta}{(1+\alpha t_0)} \tag{18}$$

$$q_0 = \left(\frac{1}{\beta}+1\right) \tag{19}$$

Solving equations (18) and (19) for $\alpha$ , $\beta$

$$\beta = \frac{-1}{1+q_0} \tag{20}$$

$$\alpha = \frac{H_0(1+q_0)}{[1-t_0H_0(1+q_0)]} \tag{21}$$

## 4. DETERMINATION OF COSMOLOGICAL PARAMETERS

### i) Cosmological constant (Λ)

From equations (14)

$$\dot{\Lambda} = -Q - nH(n-1)\left[\dot{H} + (n+3)H^2\right]$$

$$\Rightarrow \dot{\Lambda} = -Q - (n^2-n)H\dot{H} + (n^2-n)(n+3)H^3$$

$$\Rightarrow \Lambda = C - Qt - (n^2-n)\int H\dot{H} - (n^2-n)(n+3)\int H^3 dt$$

From equation (16)

$$\Rightarrow \Lambda = C - Qt - (n^2-n)\frac{H^2}{2} + (n^2-n)(n+3)\int\left(\frac{\alpha\beta}{(1+\alpha t)}\right)^3 dt$$

$$\Rightarrow \Lambda = C - Qt - \frac{(n^2-n)\alpha^2\beta^2}{2(1+\alpha t)^2} - \frac{(n^3+2n^2-3n)\alpha^2\beta^2}{2(1+\alpha t)^2}$$

$$\Rightarrow \Lambda = C - Qt - \frac{(n^3+3n^2-4n)\alpha^2\beta^2}{2(1+\alpha t)^2} \tag{22}$$

### ii) Energy density (ρ)

From equations (7) and (16)

$$\rho = -\Lambda + \frac{3(n+1)\alpha^2\beta^2}{(1+\alpha t)^2}$$

$$\Rightarrow \rho = -\mathrm{C} + \mathrm{Qt} + \frac{(n^3+3n^2+2n+6)\alpha^2\beta^2}{2(1+\alpha t)^2}$$

$$C = -\rho_0 + \mathrm{Q}t_0 + \frac{(n^3+3n^2+2n+6)\alpha^2\beta^2}{2(1+\alpha t_0)^2}$$

If $Q = 0$ , $\rho_0 = 9.83 \times 10^{-27}$ and $t_0 = 13.7 \times 10^9$

For $n = -500$ , $C = 1.007 \times 10^{-9}$

For $n = -600$ , $C = 1.74 \times 10^{-9}$

For $n = -600$ , $C = 2.77 \times 10^{-9}$

### iii) Equation of state parameter ($\omega$ )

$$\omega = \frac{\Lambda - (n+2)\dot{H} + (n^2+2n+3)H^2}{\rho}$$

From equation (16) $H = \frac{-\alpha\beta}{(1+\alpha t)}$

$$\omega = \frac{\left[C - Qt - \frac{(n^3+5n^2+6)\alpha^2\beta^2}{2(1+\alpha t)^2} + \frac{(n+2)\alpha^2\beta}{(1+\alpha t)^2}\right]}{\left[-C + Qt + \frac{(n^3+3n^2-n+3)\alpha^2\beta^2}{2(1+\alpha t)^2}\right]}$$

$$\Rightarrow \omega = \frac{\left[2(C-Qt)(1+\alpha t)^2 - (n^3+5n^2+6)\alpha^2\beta^2 + 2(n+2)\alpha^2\beta\right]}{\left[2(Qt-C)(1+\alpha t)^2 + (n^3+3n^2-n+3)\alpha^2\beta^2\right]}$$

### iv) Skewness parameter ( $\delta$ )

$$\delta = \frac{\left[(n-1)\dot{H} + (n^2+2n-3)H^2\right]}{\rho}$$

From equation (16)

$$\Rightarrow \delta = \frac{\left[2(n^2+2n-3)\alpha^2\beta^2 - 2(n-1)\alpha^2\beta\right]}{\left[2(Qt-C)(1+\alpha t)^2 + (n^3+3n^2-n+3)\alpha^2\beta^2\right]}$$

### v) Anisotropy factor: $\left(\frac{\sigma^2}{\theta}\right)$

$$\sigma^2 = \frac{3}{8}\left(\frac{\dot{a}}{a} - \frac{\dot{A}}{A}\right)^2 = \frac{3}{8}(1-n)^2H^2$$

$$\theta = 3\frac{\dot{a}}{a} + \frac{\dot{A}}{A} = (n+3)H$$

$$\frac{\sigma^2}{\theta} = \frac{3(1-n)^2H}{8(n+3)}$$

$$\Rightarrow \frac{\sigma^2}{\theta} = \frac{-3(1-n)^2\alpha\beta}{8(n+3)(1+\alpha t)}$$

### vi) Constants C and Q:

$$C = -\rho_0 + Qt_0 + \frac{(n^3 + 3n^2 + 2n + 6)\alpha^2\beta^2}{2(1 + \alpha t_0)^2}$$

$$Q = \frac{1}{t_0}\left\{C + \rho_0 - \frac{(n^3 + 3n^2 + 2n + 6)\alpha^2\beta^2}{2(1 + \alpha t_0)^2}\right\}$$

For $Q = 0$ , $n = -500$ , $t_0 = 13.7 \times 10^9$

$\rho_0 = 9.83 \times 10^{-27}$ ,

$\alpha = -7.3 \times 10^{-11}$ ,

$\beta = -2.2222$ ,

$C = 6.7032 \times 10^{-6}$ at $n = -500$

$C = 1.159 \times 10^{-5}$ at $n = -600$

$C = 1.84 \times 10^{-5}$ at $n = -700$

## FIGURES:

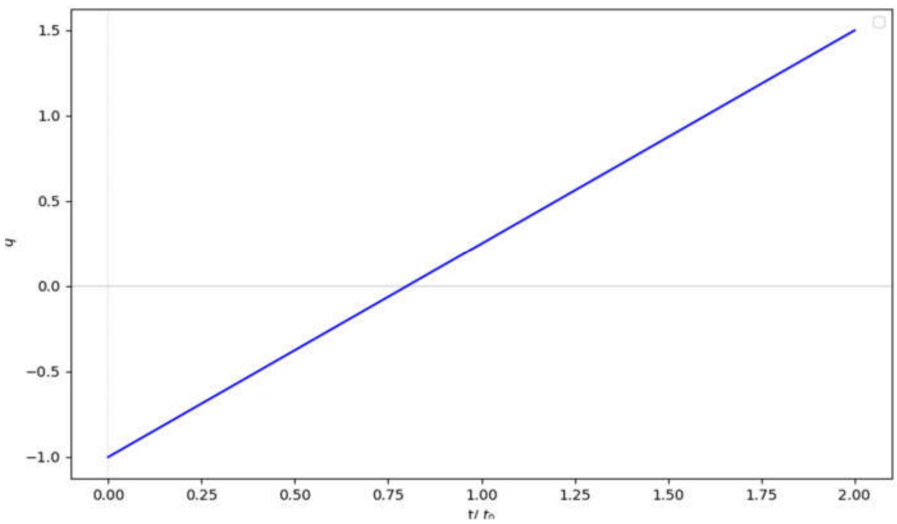


Figure 1. Deceleration parameter $(q)$ versus time.

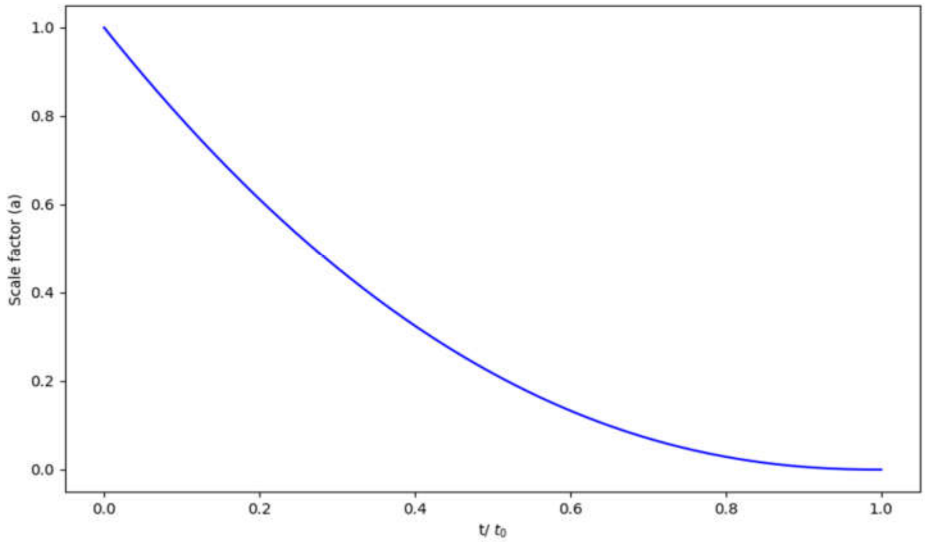


Figure 2: The scale factor versus time.

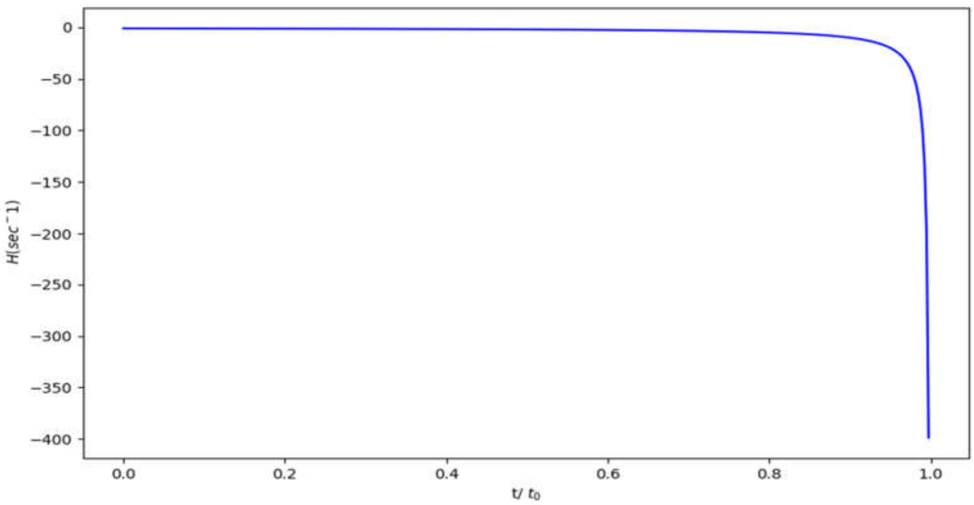


Figure 3: Hubble parameter verses time.

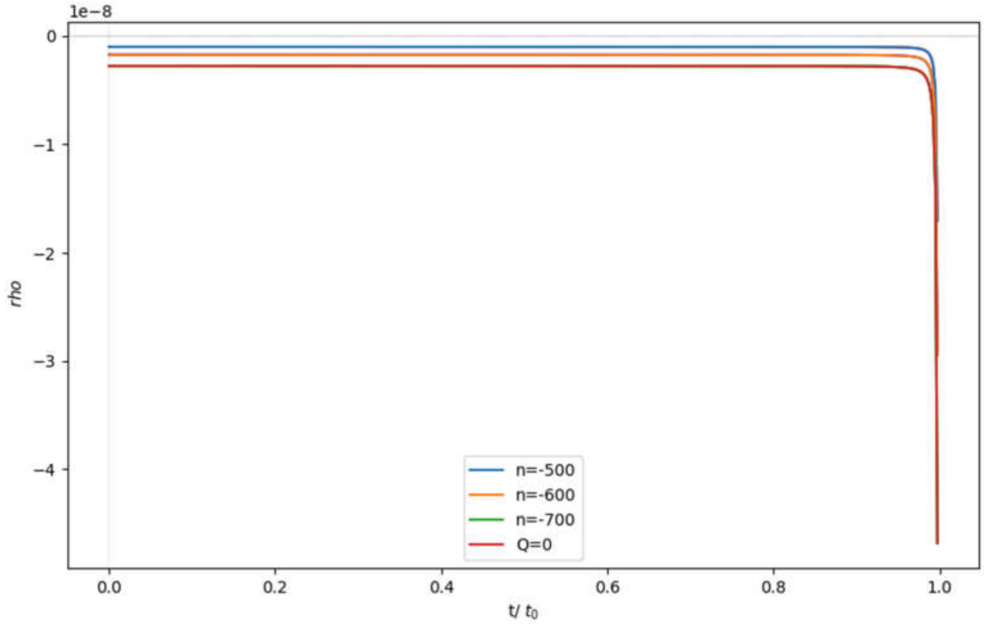


Figure 4: The energy density ($\rho$) verses time.
(for $n = -500,\ n = -600, n = -700$)

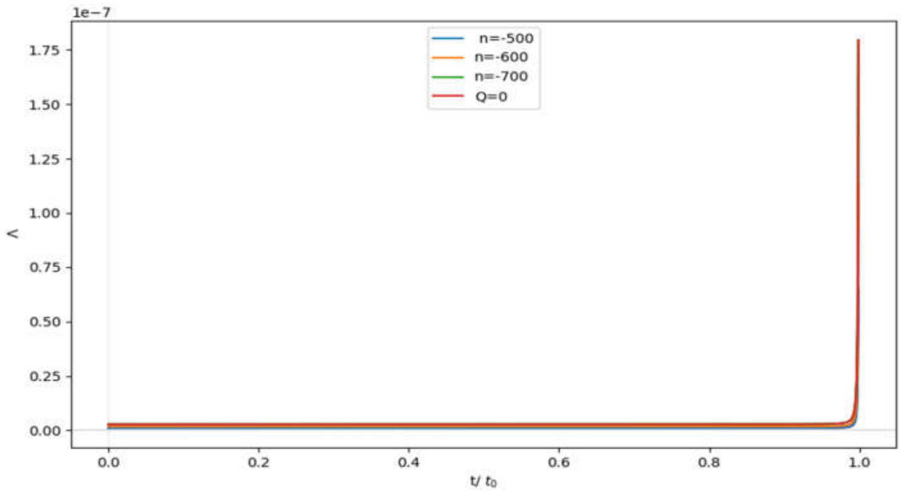


Figure 5: The dynamical cosmological parameter ($\Lambda$) verses time
(for $n = -500,\ n = -600, n = -700$)

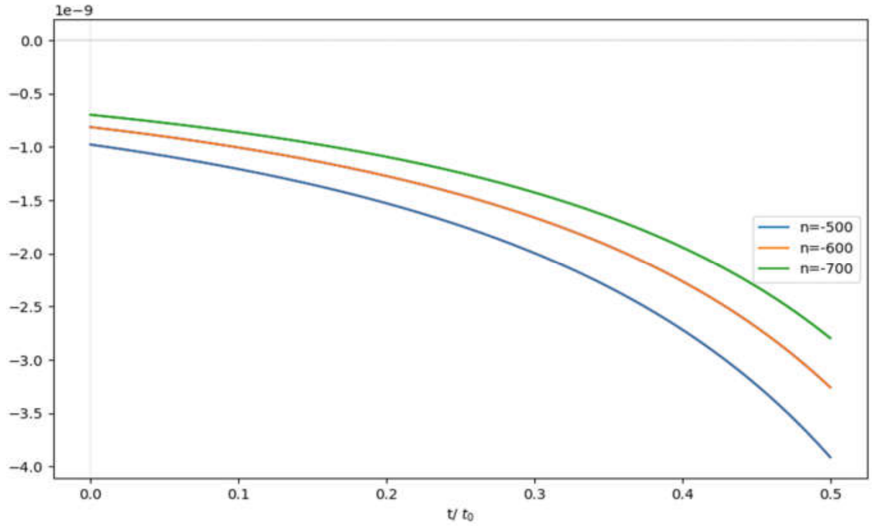


Figure 6: The skewness parameter $(\delta)$ verses time.
(for $n = -500,\ n = -600, n = \ -700$)

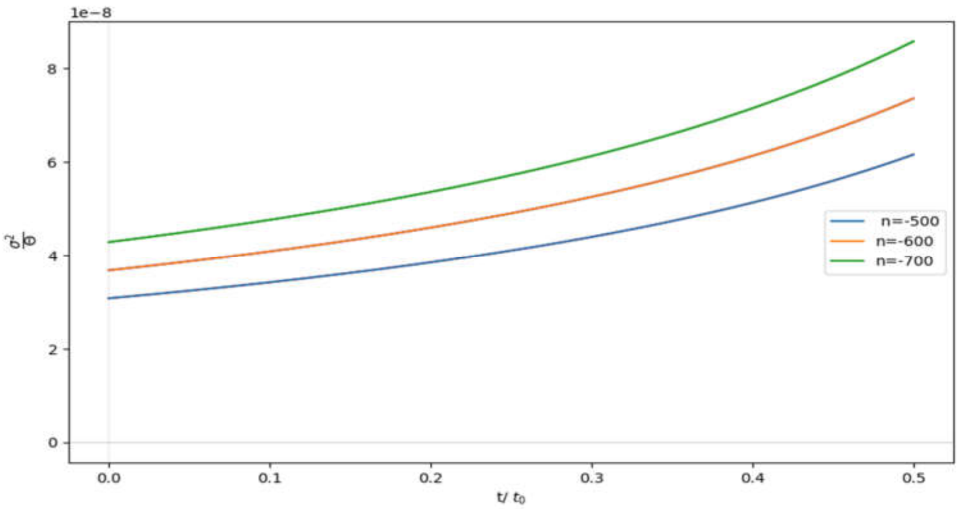


Figure 7: The anisotropy parameter $\left(\frac{\sigma^2}{\theta}\right)$ verses time.
(for $n = -500,\ n = -600, n = \ -700$)

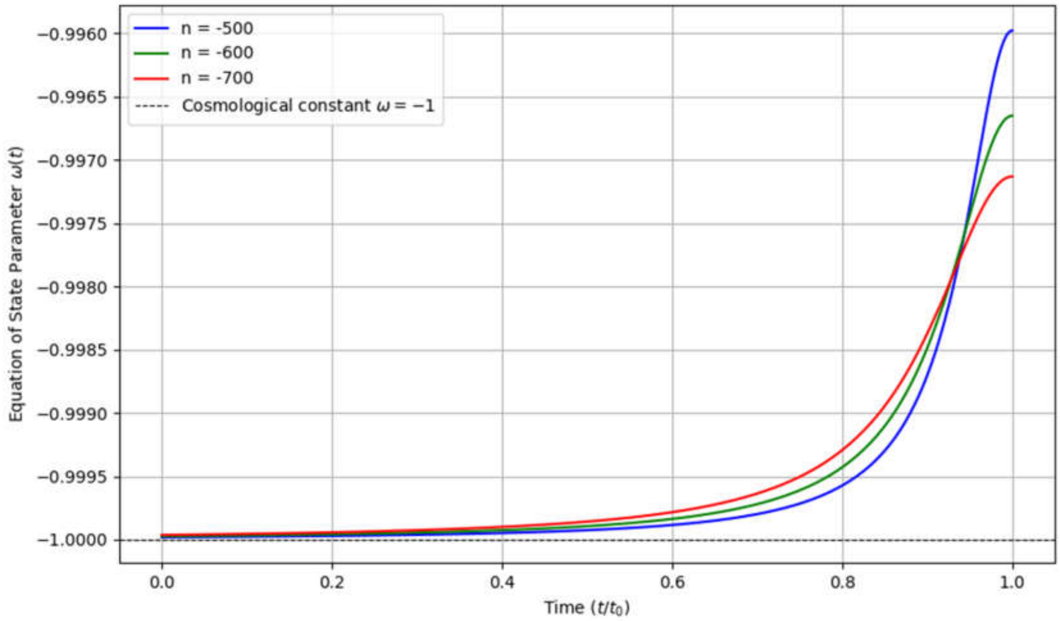


Figure 8: The Eos parameter $(\omega)$ verses time.

(for $n = -500,\ n = -600, n = \ -700$)

## 5. Conclusion

This study investigates the evolution of the universe within an anisotropic Kaluza-Klein space-time framework, featuring a time-dependent cosmological constant $(\Lambda)$. An empirical scale factor was chosen to produce a deceleration parameter that transitions from positive to negative, indicative of a shift from decelerated to accelerated expansion, aligning with current observational findings. To improve the model's predictive power, the constants $\alpha, \beta$, and $C$ were determined using established values of the Hubble constant ( H_0 ), the current deceleration parameter $(q_0)$, and the present energy density ( $\rho_0$). Due to the absence of experimental data, the parameter ( $n$ ) could not be fixed; however, it must satisfy $(n < 0\ )$ to produce physically meaningful results. As shown in Figure-4, the slopes of the energy density ( $\rho_0$). curves vary for different ( $n$ ) values at all $(\frac{t}{t_0}\ )$, but precise determination of ( $n$ ) would require an observational estimate of ( $\frac{d\rho}{dt}$ ) at the current time ( $t = t_0$). An important outcome of this research is that the dynamical $\Lambda$ remains negative but becomes less so over time, approaching a value around $(-9.83\ \times\ 10^{-27})$, which is numerically equal to the current energy density. The evolution of the equation of state parameter $(\omega)$ suggests that the universe has been in a phantom dark energy regime ( $\omega < -1$ ) since the early universe and is gradually approaching a vacuum-dominated phase ( $\omega < -1$ ). Additionally, the anisotropy parameter $(\frac{\sigma^2}{\theta} \to 0)$ as ( $\frac{t}{t_0} \to \infty$), indicating a universe becoming increasingly isotropic. While equation (11)—derived from the field equations—was not utilized here, the authors note that setting $(\omega = 1\ )$ in that equation would yield an expression for $(\rho(t)\ )$, with its constant of integration determined by the present-day density. Future work will explore this approach further to refine the evolution of other cosmological quantities.

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