On Transformations Preserving Visibility of Lattice Points

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Abstract

We study the notion of visibility of lattice points from the origin in the integer lattice Z^2, where a point (x, y) is said to be visible if $\gcd(x, y)$. Building on the classical visibility criterion, we investigate classes of transformations that preserve visibility. In particular, we identify and analyse families of nonlinear maps under which visibility is invariant, including polynomial and power-type transformations. We establish sufficient arithmetic conditions guaranteeing that if a lattice point is visible, then its image under these transformations remains visible. Special attention is given to exceptional boundary cases and their geometric interpretation. These results extend the classical theory beyond unimodular linear transformations and provide a unified framework for understanding visibility-preserving behaviour in the integer lattice.

Keywords: Greatest common divisor, Visibility of Lattice Point, Linear and non-Linear Transformation.

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1 INTRODUCTION

The geometry of lattice points has played a central role in number theory and discrete geometry since the time of Gauss. One of the most classical notions in this area is that of visibility from the origin. A lattice point $(x, y) \in Z^2$ is said to be visible from the origin if the line segment joining (0.0) and (x, y) contains no other lattice point in its interior. It is well known that this geometric condition is equivalent to the arithmetic condition $\gcd(x, y) = 1$, a fact discussed in standard number theory texts such as Hardy and Wright **[14],** Apostol **[7],** and Niven et al. **[8].**

The study of visible lattice points is closely connected with several fundamental topics in number theory, including coprimality, arithmetic functions, and Diophantine approximation. In particular, the asymptotic density of visible lattice points in Z^2 is known to be $6/_{\pi^2}$ a result that follows from properties of the Mobius function and appears in classical analytic number theory literature **[7, 8].** From a geometric perspective, visible lattice points are naturally related to primitive vectors and the structure of rational slopes, as explored in discrete and computational geometry **[13].** Beyond counting problems, visibility has been studied from probabilistic and dynamical viewpoints. The distribution of visible lattice points in expanding regions and their angular distribution have been investigated in detail by Boca and Zaharescu **[11],** highlighting deep connections between visibility, uniform distribution, and ergodic theory. These works emphasize that visibility is not merely a local arithmetic condition but reflects global geometric and statistical structure.

A natural question arising from this theory concerns the behaviour of visibility under transformations of lattice points. Linear transformations with determinant ± 1, known as unimodular transformations, preserve visibility since they preserve co-primality. Such transformations form the classical symmetry group of the lattice and are well understood **[13, 14].** However, many nonlinear transformations fail to preserve visibility, and comparatively little attention has been paid to identifying nonlinear maps that do preserve this property. Recent interest in discrete dynamical systems and arithmetic geometry motivates the systematic study of visibility-preserving transformations beyond the linear setting. Understanding how co-primality behaves under nonlinear operations such as powering, coordinate mixing, and product constructions is essential for extending classical lattice geometry into new arithmetic contexts. Despite scattered examples appearing implicitly in the literature, a unified treatment of such transformations appears to be missing.

1.1 Historical background of lattice point visibility

The historical record of the original authorship of the concept of visibility of lattice point and their results is inaccurately described on a number of occasions in the literature. Originally, the question on the probability of two random integers being co-prime was raised in 1881 by Cesaro **[2]**. Two years later, he and Sylvester independently proved the result **[3, 4]** respectively. Earlier in **1849**, Dirichlet proved a slightly weaker form of the result **[1].** The generalization to k co prime integers with k > 2 was presented again by Cesaro in **1884 [5].** This result was apparently proven independently in 1900 by Lehmer **[6].** Later on, the study lattice point visibility is a classical theme in number theory and discrete geometry. Visible lattice points are also widely concerned in number theory, for example their density (or their numbers) in certain regions **[7].** This concept has been studied and used in various areas such as computer science, integer optimization and theoretical physics **[9].** More interestingly Wataru Takeda showed the exact order of the number of lattice points visible from the origin **[15].** In **[10, 12],** a new method was developed to study the location and well distributed of the lattice point around the circles centre at the origin and also associate the lattice points with Gaussian integer. Additionally, for the fixed-point b ∈ N, the lattice point $(r, s) \in N \times$

N is said to b - visible (or) b - invisible based on the lattice point lies on the graph $f(x) = ax^b$ for some $a \in Q$ and the proportion of b-visible integer lattice points is given by $1/\zeta(b+1)$, where $\zeta(s)$ denotes the Riemann zeta function **[16].** The above concept of visibility can be generalized, namely, to consider the visibility along certain type of curves and its distributions. See some recent and wide works in this direction of simultaneous visibility and its Ergodic-theoretical viewpoint **[17, 18].**

1.2 Objectives of the Study

- To identify transformations on Z^2 that preserve visibility of lattice points from the origin.
- To characterize visibility preservation in terms of arithmetic invariants, particularly greatest common divisors.
- To analyse the effect of non-linear, non-unimodular transformations on lattice visibility.
- To construct and study Fermat-type and power-type transformations preserving visibility.
- To compare visibility-preserving transformations with classical unimodular maps.
- To extend the classical theory of lattice visibility to broader arithmetic settings.

1.3 Motivation of the Study

- Visibility of lattice points from the origin, characterized by $\gcd(x, y) = 1$, is classically known to be preserved under unimodular linear transformations.
- A systematic investigation of visibility under non-linear and non-unimodular transformations is largely absent from the existing literature.
- Recent advances in lattice geometry, arithmetic combinatorics and computational number theory emphasize the need to understand how arithmetic properties of lattice points behave under broader classes of transformations.
- This work is motivated by the fundamental question:
  Which transformations map visible lattice points to visible lattice points, and why does this preservation occur?
- Transformations such as translations, diagonal shifts, polynomial maps, power maps and Fermat-type constructions arise naturally in number theory but have not been studied from the perspective of lattice visibility.
- Our analysis reveals that visibility preservation is frequently governed by hidden gcd invariants, rather than linear structure alone, indicating a deeper arithmetic mechanism.

- By identifying explicit families of transformations that preserve visibility, this work extends the classical theory beyond unimodular maps and broadens the conceptual framework of lattice visibility.
- Ultimately, this study aims to reinterpret lattice visibility as a transformation-invariant arithmetic property rather than a static condition tied solely to the origin.

2 PRELIMINARIES

In this section, we collectively provide definitions and basic results that will be used throughout the paper. All integers are assumed to be nonzero unless stated otherwise.

Definition 2.1. [7] Let α and β be given integers, with at least one of them different from zero. The greatest common divisor of α and β, denoted by $\gcd(\alpha, \beta)$is the positive integer r satisfying the following:
(i) r | α and r | β
(ii) If c | α and c | β then c ≤ r.

Definition 2.2. [7] The greatest common divisor of two integers m and n is the largest integer that divides them both: $\gcd(x, y) = \max\left\{k : \frac{x}{k} \text{ and } \frac{y}{k}\right\}$. (Or) we simply say, two integers x and y, not both which are zero are said to be relatively prime whenever $\gcd(x, y) = 1$.

For example, gcd (12; 18) = 6. This is a familiar notion, because it's the common factor that fourth graders learn to take out of a fraction x/y when reducing it to lowest terms: $12/18 = (12/6)(18/6) = 2/3$. Notice that if y > 0 we have $\gcd(0, y) = y$, because any positive number divides 0, and because y is the largest divisor of itself. The value of gcd (0,0) is undefined.

Definition 2.3. [7] A lattice point is any point in a coordinate system (such as 2D or 3D) where each coordinate is an integer value that is it can be written as (x, y) or (x, y, z) where $x, y, z \in Z$ (the set of integers).

2.1 Properties of Greatest Common Divisor and Notation

We recall several standard properties of Greatest Common Divisor that will be used repeatedly.
Let a, b, c ∈ Z.
The following are the fundamental properties of the greatest common divisor gcd (a, b):
(i) **Symmetry:** gcd (a, b) = gcd (b, a).
(ii) **Non-negativity:** gcd (a, b) ≥ 1 unless a = b = 0.

(iii) **Normalization:** gcd (a, 0) = |a|, gcd (0, b) = |b|.

(iv) **Euclidean algorithm property:** gcd (a, b) = gcd (b, a − b), more generally gcd (a, b) = gcd (a, b −ka) where k ∈ Z.

(v) **Bezout identity (linear combination property):** gcd (a, b) = min {|ax+by|: x, y ∈ Z}, equivalently $\exists\, x, y \in Z$ such that ax + by = gcd (a, b).

(vi) **Co primality criterion:** gcd (a, b) = 1 ⇔ ∃ x, y ∈ Z such that ax + by = 1.

(vii) **Extension to multiple integers:** gcd (a, b, c) = gcd (gcd (a, b), c).

(viii) **Multiplicative property:** If d = gcd (a, b), then gcd $\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.

(ix) **Invariance under sign:** gcd (a, b) = gcd (|a|, |b|).

For the convenience of the reader, we summarize the main notation used throughout the paper.

Symbol	Meaning
Z^2	Integer lattice in the plane.
(x, y)	A lattice point with integer coordinates.
gcd (a, b)	Greatest common divisor of a and b.
Visible point	A lattice point with gcd (x, y) = 1.
$T: Z^2 \to Z^2$	A lattice transformation.
Unimodular transformation	Linear map with determinant ± 1.
Nonlinear transformation	Map not linear in both coordinates.
g_n	$(2k)^{2n} + 1$, Fermat-type sequence.
(g_m, g_n)	Fermat-type lattice point.

2.2 Lattice Points and Visibility

A lattice point in the plane is a point (x, y) with x, y ∈ Z. The set of all lattice points is denoted by Z^2

Definition 2.2.1. A lattice point (x, y) ∈ Z^2 is said to be visible from the origin if the line segment joining (0, 0) and (x, y) contains no other lattice point in its interior.

Theorem 2.4 (Visibility Criterion). A lattice point (x, y) ∈ Z^2 is visible from the origin if and only if gcd (x, y) = 1.

Proof. Suppose gcd (x, y) = d > 1. Then d divides both x and y, so we can write x = d·x_1 and y = d·y_1 for some integers x_1, y_1. Thus, the point (x_1, y_1) = (x/d, y/d) is a lattice point. Since d > 1, this point lies strictly between (0,0) and (x, y) on the same line. Hence (x, y) is not visible from the origin.

Conversely, suppose gcd (x, y) = 1, and assume for contradiction that there exists a lattice point (a, b) strictly between (0,0) and (x, y). Then (a, b) = t (x, y) for some real number 0 < t < 1. Since a and b are integers, it follows that t = a/x = b/y. This implies that x divides ay and y divides bx, leading to a common divisor greater than 1 dividing both x and y, which contradicts gcd (x, y) = 1. Therefore, no such intermediate lattice point exists, and (x, y) is visible from the origin

2.3 Visibility-Preserving Transformations

We now formalize the central notion of this paper.

Definition 2.3.1. A transformation $T: Z^2 \to Z^2$ is said to preserve visibility if gcd (x, y) = 1 ⇒ gcd (T_1 (x, y), T_2 (x, y)) = 1, where T (x, y) = (T_1(x, y), T_2(x, y)).

Unimodular linear transformations provide classical examples.

Example 2.3.2. If T (x, y) = (ax + by, cx + dy) with ad − bc = ± 1, then T preserves visibility.

However, most nonlinear transformations do not preserve visibility. The goal of this paper is to identify structured nonlinear transformations that do.

2.4 Linear and Unimodular Transformations

We begin by recalling basic notions from linear algebra that play a central role in classical visibility theory.

Definition 2.4.1. A map $T: Z^2 \to Z^2$ is called a linear transformation if it is of the form T (x, y) = (ax + by, cx + dy), where a, b, c, d ∈ Z.

Such transformations correspond to multiplication by a 2 × 2 integer matrix.

Definition 2.4.2. A linear transformation T (x, y) = (ax + by, cx + dy) is called unimodular if

$$det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = \pm 1.$$

Unimodular transformations are precisely the invertible linear transformations over Z. They preserve co-primality and therefore preserve visibility of lattice points. This fact makes unimodular transformations the classical symmetry group of visible lattice points.

2.5 Nonlinear Transformations

Not all transformations preserving lattice structure are linear.

Definition 2.5.1. A map $T: Z^2 \to Z^2$ is called a nonlinear transformation if at least one coordinate function of T (x, y) is nonlinear in x and y.

Typical examples include power maps, product-based maps, and mixed-coordinate maps such as $(x, y) \to (x^k, y^k)$, $(x, y) \to (x, yz)$, $(x, y) \to (x + y,\ y)$.

Unlike unimodular linear transformations, nonlinear transformations do not generally preserve visibility. Determining when such maps preserve co-primality is a central theme of this paper.

Example 2.5.2. ***Failure of Visibility*:** Not every reasonable-looking transformation preserves visibility. For example, the map (x, y) → (x^2, xy) does not preserve visibility in general. Throughout the paper, we emphasize the importance of identifying precise conditions under which visibility is maintained.

These preliminaries provide the foundation for the results developed in the subsequent sections.

3 PRESERVATIONS OF VISIBILITY OF LATTICE POINT UNDER TRANSFORMATION

Theorem 3.1 (Unimodular Invariance of Visibility). If the lattice point (x, y) ∈ Z^2 is visible from the origin then (x', y') = (ax+by, cx+dy) where a, b, c, d ∈ Z satisfy ad−bc = ± 1 is also visible from the origin.

Proof. Let (x, y) is visible from the origin, we have gcd (x, y) = 1. Suppose, for contradiction, that (x′, y′) is not visible from the origin. Then there exists an integer k > 1 such that k | x′ and k | y′. That is,

$$k \mid (ax + by) \text{ and } k \mid (cx + dy).$$

Now, consider the integer linear combinations

$$d\,(ax + by) - b\,(cx + dy) = (ad - bc)\,x,$$

$$-a\,(cx + dy) + c\,(ax + by) = (ad - bc)\,y.$$

Since k divides both ax + by and cx + dy, it follows that

$$k \mid (ad - bc)\,x \text{ and } k \mid (ad - bc)\,y.$$

Because ad−bc = ± 1, we obtain k | x and k | y. This implies k | gcd (x, y), which contradicts gcd (x, y) = 1. Therefore, gcd (x′, y′) = 1, and hence (x′, y′) is visible from the origin.

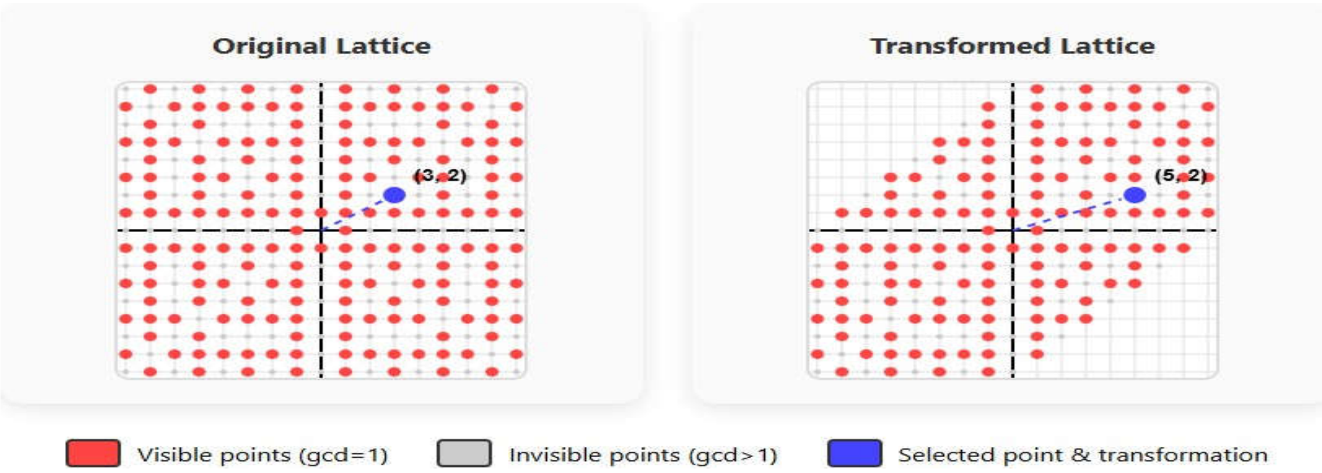


Figure 1: TRANSFORMATION STRUCTURE Of VISIBILITY Of LATTICE POINTS.

Corollary 3.1.1. If the lattice point (x, y) ∈ Z^2 be visible from the origin then lattice point (x, x-y) is also visible from the origin.

Proof. Consider the linear transformation (x', y') = (x, x−y). This transformation is induced by the integer matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

A direct computation shows that det(A) = (1) (−1) − (0)(1) = −1. Hence, A is unimodular. By (Thm.3.1), unimodular transformations preserve visibility of lattice points. Therefore, if (x, y) is visible from the origin then (x, x − y) is also visible from the origin.

Corollary 3.1.2. If the lattice point (x, y) ∈ Z^2 be visible from the origin then lattice point (x+y, y) is also visible from the origin.

Proof. Consider the linear transformation (x', y') = (x + y, y), which is induced by the integer matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The determinant of A is det(A) = (1)(1)−(1)(0) = 1. Hence, A is unimodular. By (Thm.3.1), unimodular transformations preserve visibility of lattice points. Therefore, if (x, y) is visible from origin, then (x+y, y) is also visible from the origin.

Corollary 3.1.3. If the lattice point (x, y) ∈ Z^2 be visible from the origin for any integer k ∈ Z, then the lattice points (x + ky, y) and (x, y + kx) are also visible from the origin.

Proof. Consider the linear transformations (x', y') = (x + ky, y) and (x', y') = (x, y + kx) are induced respectively by the matrices

$$A_1 = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 0 \\ k & -1 \end{pmatrix}$$

A direct computation shows that det (A_1) = det (A_2) = 1. Thus, both matrices are unimodular. By (Thm.3.1), unimodular transformations preserve visibility. Hence, both (x + ky, y) and (x, y + kx) are visible from the origin whenever (x, y) is visible.

Theorem 3.2 (Visibility Preserved Under Power Maps). Let the lattice point (x, y) ∈ Z^2 \{(0, 0)} and let k ≥ 1 be an integer. Then (x, y) is visible from the origin if and only if $(x^k, y^k)$is visible from the origin.

Proof. Suppose a lattice point (a, b) ∈ Z^2 is not visible from the origin if and only if there exists an integer m ≥ 2 and a lattice point (u, v) ∈ Z^2 such that (a, b) = m (u, v). Equivalently, the line segment from (0, 0) to (a, b) contains a lattice point other than the endpoints.

(⇒) Necessity: Assume (x, y) is visible from the origin. Suppose, for contradiction, that(x^k, y^k) is not visible. Then there exists an integer m ≥ 2 and a lattice point (u, v) such that (x^k, y^k)= m (u, v). Hence m divides both x^k *and* y^k. Let p be any prime divisor of m. Then p | x^k and p | y^k which implies p | x and p | y. Thus, (x, y) = $p(\frac{x}{p}, \frac{y}{p})$ the lattice point $(\frac{x}{p}, \frac{y}{p})$ lies strictly between (0, 0) and (x, y) on the same line. This contradicts the visibility of (x, y). Therefore (x^k, y^k) must be visible from the origin.

(⇐) Sufficiency: Assume $(x^k, y^k)$is visible from the origin. Suppose, for contradiction, that (x, y) is not visible. Then there exists an integer m ≥ 2 and a lattice point (u, v) such that (x, y) = m (u, v). Raising both coordinates to the kth power yields, (x^k, y^k)= $m^k(u^k, v^k)$, which shows that $(x^k, y^k)$is a nontrivial integer multiple of another lattice point. Hence the line segment from (0, 0) to (x^k, y^k)contains lattice points other than the endpoints, contradicting its visibility. Hence (x, y) is visible from the origin if and only if (x^k, y^k) is visible from the origin.

Example 3.2.1 (Visibility Preserved Under Squaring). Let the lattice point (x, y) ∈ Z^2 \{(0, 0)}. Then (x, y) is visible from the origin if and only if (x^2, y^2) is visible from the origin.

Proof. (⇒) Assume (x, y) is visible from the origin. Suppose, for contradiction, that (x^2, y^2) is not visible. Then there exists an integer m ≥ 2 and a lattice point (u, v) such that (x^2, y^2) = m (u, v). Hence m divides both x^2 and y^2. Let p be any prime divisor of m. Then p | x^2 and p | y^2, which implies p | x and p | y. Consequently, (x, y) = p (x/p, y/p), and the lattice point (x/p, y/p) lies strictly between (0,0) and (x, y) on the same line. This contradicts the visibility of (x, y). Therefore (x^2, y^2) must be visible from the origin.

(⇐) Assume (x^2, y^2) is visible from the origin. Suppose, for contradiction, that (x, y) is not visible. Then there exists an integer m ≥ 2 and a lattice point (u, v) such that (x, y) = m (u, v). Squaring both coordinate's yields, (x^2, y^2) = m^2 (u^2, v^2), showing that (x^2, y^2) is a nontrivial integer multiple of another lattice point. Thus, the line segment from (0,0) to (x^2, y^2) contains intermediate lattice points, contradicting its visibility. Hence (x, y) is visible from the origin if and only if (x^2, y^2) is visible.

Figure 2: SQUARED TRANSFORMATION STRUCTURE Of VISIBILITY Of LATTICE POINTS

Theorem 3.3 (General Parity–Controlled Visibility). Let the lattice point (x, y) is visible from the origin and let k ≥ 2 be an even integer, x ∈ Z then the lattice point (x + 1, x^k + 1) is visible from the origin if and only if x is even.

Proof. Let the lattice point (x, y) is visible from the origin. Now, we compute gcd (x + 1, x^k + 1). By applying the Euclidean algorithm, we obtain

$$\gcd(x + 1, x^k + 1) = \gcd(x + 1, x^k + 1 - x^{k-1}(x + 1)) = \gcd(x + 1, x^{k-11} - 1).$$

Since k is even, k −1 is odd. Now, we define Q(x) = $x^{k-2} - x^{k-3} + x^{k-4} - \cdots - x + 1$. A direct expansion shows that (x + 1) Q(x) = x^{k-1}+ 1, since all intermediate terms cancel pairwise. Consequently,

$$(x^{k-1} - 1) - (x + 1)\,Q(x) = -2.$$

Let d be any common divisor of x + 1 and x^{k-1} − 1. Then

$$d \mid (x + 1) \text{ and } d \mid (x^{k-1} - 1) \Rightarrow d \mid [(x^{k-1} - 1) - (x + 1)\,Q(x)] \Rightarrow d \mid 2.$$

Therefore, any common divisor of x + 1 and x^{k-1}− 1 must divide 2, and hence

$$\gcd(x + 1, x^{k-1} - 1) = \gcd(x + 1, 2).$$

Therefore gcd (x + 1, x^{k-1}− 1) = gcd (x + 1, 2). If x is even, then x + 1 is odd and gcd (x + 1, 2) = 1, so the point is visible from the origin. If x is odd, then x + 1 is even and gcd (x + 1,2) = 2, so the point is not visible.

Example 3.3.1 (Parity–controlled visibility). If the lattice point (x, y) is visible form the origin and x ∈ Z be an even integer then the lattice point (x + 1, x^2 + 1) is also visible from the origin.

Proof. Let the lattice point (x, y) is visible from the origin (i.e, gcd (x, y) = 1). Hence it suffices to compute gcd (x + 1, x^2 + 1).

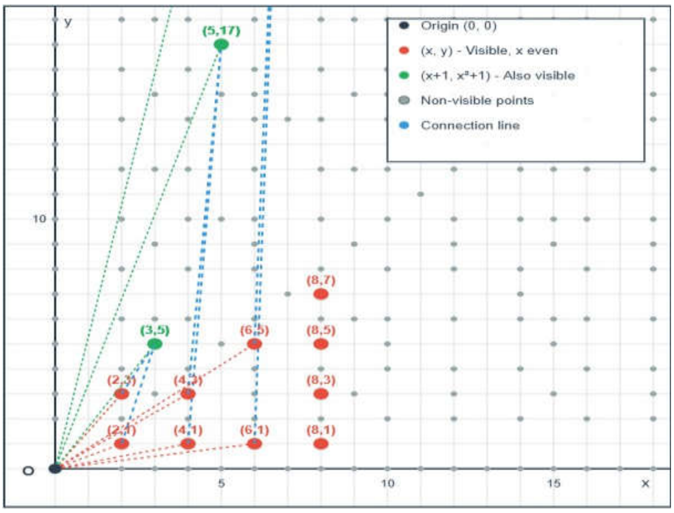
Now applying the Euclidean algorithm, we obtain

$$\begin{aligned}\gcd(x + 1, x^2 + 1) &= \gcd(x + 1, x^2 + 1 - x(x + 1)) \text{ [by gcd (a, b) = gcd (a, b − ka)]}\\ &= \gcd(x + 1, 1 - x)\\ &= \gcd(x + 1, x - 1) \text{ [by gcd (a, b) = gcd (a, −b)]}\\ &= \gcd(x + 1 - (x - 1), x - 1) \text{ [by gcd (a, b) = gcd (a − kb, b)]}\\ &= \gcd(2, x - 1).\end{aligned}$$

Since x is even, x − 1 is odd, and therefore gcd (2, x − 1) = 1. It follows that gcd (x + 1, x^2+ 1) = 1, and hence the point (x + 1, x^2+ 1) is visible from the origin.

Remark.

This example shows that visibility can be preserved under specific nonlinear transformations controlled by parity conditions.



Examples Demonstrating the Statement:

x (even)	(x, y)	(x+1, x²+1)	gcd verification
2	(2, 1)	(3, 5)	gcd(3,5) = 1 ✓
2	(2, 3)	(3, 5)	gcd(3,5) = 1 ✓
4	(4, 1)	(5, 17)	gcd(5,17) = 1 ✓
4	(4, 3)	(5, 17)	gcd(5,17) = 1 ✓
6	(6, 1)	(7, 37)	gcd(7,37) = 1 ✓
6	(6, 5)	(7, 37)	gcd(7,37) = 1 ✓
8	(8, 1)	(9, 65)	gcd(9,65) = 1 ✓
8	(8, 3)	(9, 65)	gcd(9,65) = 1 ✓

(a) Graphical View **(b) Verification Table**

Figure 3: PARITY CONTROLLED TRANSFORMATION STRUCTURE Of VISIBILITY Of LATTICE POINTS.

Theorem 3.4 (Visibility under multiplicative projection). Let $x, y, z \in Z$. If the lattice points *(x, z)* and *(x, z)* are visible from the origin, then the lattice point *(x, yz)* is also visible from the origin.

Proof. Let the lattice point *(x, y)* and *(x, z)* is Visible from the origin is equivalent to gcd *(x, y)* = 1, and gcd *(x, z)* = 1. Assume, for contradiction, that *(x, yz)* is not visible from the origin. Then gcd *(x, yz)* ≠ 1. Hence there exists a prime p such that *p | x and p | yz.* Since *p | yz*, we must have *p | y* or p | *z*. If *p | y*, then *p | x* and *p | y*, which implies gcd (*x, y*) ≠ 1,, contradicting the visibility of (*x, y*). If *p | z*, then *p | x* and *p | z*, which implies gcd (*x, z*) ≠ 1, contradicting the visibility of (*x, z*). Both cases lead to a contradiction. Therefore, no such prime *p* exists, and hence gcd (*x, yz*) = 1. Thus, the lattice point (*x, yz*) is visible from the origin.

Case 1: If $x \neq 0$. By the visibility argument given above, it follows that gcd (*x, yz*) = 1, and hence the lattice point (*x, yz*) is visible from the origin.

Case 2: If $x = 0$. The lattice point (0, *y*) is visible from the origin if and only if $|y| = 1$, and similarly (0, *z*) is visible if and only if $|z| = 1$. Thus $y, z \in \{\pm 1\}$. Consequently, $yz = \pm 1$, and hence (0, *yz*) = (0, ±1) is visible from the origin.

In both cases, the lattice point (*x, yz*) is visible from the origin. Therefore, the statement of the theorem holds without exception.

Corollary 3.4.1 (Visibility under finite multiplicative projection). Let $x, y_1, y_2, \dots, y_k \in \mathbb{Z}$. If each lattice point (x, y_i) is visible from the origin for $i = 1, 2, \dots, k$, then the lattice point $\left(x, \prod_{i=1}^{k} y_i\right)$ is also visible from the origin.

Proof. Visibility of (x, y_i) from the origin is equivalent to $\gcd(x, y_i) = 1$ for all $i = 1, 2, \dots, k$. Assume, for contradiction, that $\left(x, \prod_{i=1}^{k} y_i\right)$ is not visible from the origin (i.e.,$\gcd\left(x, \prod_{i=1}^{k} y_i\right) \neq 1$.) Hence there exists a prime p such that

$$p \mid x \quad \text{and} \quad p \mid \prod_{i=1}^{k} y_i.$$

Since a prime dividing a product divide at least one factor, there exists an index $j \in \{1, 2, \dots, k\}$ such that $p \mid y_j$. Thus $p \mid x$ and $p \mid y_j$, which implies $\gcd(x, y_j) \neq 1$, contradicting the visibility of (x, y_j). Therefore, no such prime p exists, and hence $\gcd\left(x, \prod_{i=1}^{k} y_i\right) = 1$. Thus, the lattice point $\left(x, \prod_{i=1}^{k} y_i\right)$ is visible from the origin. □

4 SPECIAL CASES OF TRANSFORMATION VIA VISIBILITY

4.1 Fermat-Type Lattice Points

We now introduce a special class of lattice points tailored to the visibility results proved in this paper.

Definition 4.1. Let $k \geq 1$ be a fixed integer and define the sequence

$$g_n = (2k)^{2n} + 1, n \geq 0$$

A lattice point $(x, y) \in Z^2$ is called a Fermat-type lattice point if $(x, y) = (g_m, g_n)$ for some integers $m, n \geq 0$.

This definition is specifically satisfied strong pairwise visibility property. Analogously, Fermat-type lattice points often exhibit visibility properties due to the arithmetic structure of their coordinates. In particular, when the co-ordinates arise from distinct exponent levels, co-primality arguments can be applied to establish visibility from the origin. Such lattice points provide natural examples of visibility-preserving constructions beyond linear theory.

Theorem 4.2 ***(Visibility of generalized Fermat-type lattice points).*** **Let** $k \geq 1$ be a fixed integer and define $g_n := (2k)^{2^n} + 1$ for all $n \geq 0$. Then for any integers $m, n \geq 0$ with $m \neq n$, the lattice point (g_m, g_n) is visible from the origin.

Proof. A lattice point $(x, y) \in \mathbb{Z}^2$ is visible from the origin if and only if $\gcd(x, y) = 1$. Thus, it suffices to show that $\gcd\left((2k)^{2^m} + 1, (2k)^{2^n} + 1\right) = 1$ for $m \neq n$. Assume, for contradiction, that there exists an integer $d > 1$ such that

$$d \mid (2k)^{2^m} + 1 \quad \text{and} \quad d \mid (2k)^{2^n} + 1,$$

with $m \neq n$. Without loss of generality, assume $m > n$. From the second divisibility we obtain

$$(2k)^{2^n} \equiv -1 \pmod{d}.$$

Squaring both sides yields

$$(2k)^{2^{n+1}} \equiv 1 \pmod{d}.$$

Hence the multiplicative order of $2k$ modulo d satisfies $\text{ord}_d(2k) = 2^{n+1}$. On the other hand, from

$$(2k)^{2^m} \equiv -1 \pmod{d},$$

we similarly deduce

$$(2k)^{2^{m+1}} \equiv 1 \pmod{d},$$

and therefore $\text{ord}_d(2k) = 2^{m+1}$. Since an element modulo d has a unique multiplicative order, we must have $2^{n+1} = 2^{m+1}$, which implies $m = n$, contradicting the assumption that $m \neq n$.

Thus, no integer $d > 1$ can divide both g_m and g_n, and hence $\gcd(g_m, g_n) = 1$ for $m \neq n$. Therefore, the lattice point (g_m, g_n) is visible from the origin.

Corollary 4.2.1. Let (g_m, g_n) be a Fermat-type lattice point with $m \neq n$. Then the translated point $(g_m + a, g_n + a)$ is visible from the origin for all integers a satisfying gcd $(g_m + a, g_n - g_m) = 1$.

Remark. Fermat-type numbers satisfy strong co-primality properties, which imply that the difference $g_n - g_m$ has a highly restricted prime structure. Consequently, the set of integers a for which $(g_m + a, g_n + a)$ fails to be visible is sparse. This rigidity explains why Fermat-type lattice points exhibit enhanced stability under certain nonlinear and transformation perturbations.

Example 4.5. Let (x, y) = (3, 4), for which gcd (3, 4) = 1. Here y - x = 1, so gcd (x + a, y - x) = gcd (3 + a, 1) = 1 for all a ∈ Z. Hence (x + a, y + a) is visible from the origin for every integer a.

Example 4.6. Let (x, y) = (3, 5), with gcd (3, 5) = 1 and y - x = 2. Choosing a = 1 gives (x + a, y + a) = (4, 6), gcd (4, 6) = 2, which is not visible from the origin. Thus, diagonal translations may destroy visibility even when the original point is visible.

Example 4.7. Let (x, y) = (4, 7), so gcd (4, 7) = 1 and y - x = 3. Then (x + a, y + a) is visible if and only if $3 \nmid (4 + a)$. Hence visibility holds for infinitely many integers a, and fails for infinitely many integers a.

Theorem 4.8.(Visibility Under Diagonal Translation) Let (x, y) ∈ Z^2 with a, b ∈ Z. The translated point (x + a, y + b) is visible from the origin if and only if gcd (x + a, y - x + b − a) = 1.

Proof. Let assume that the translated point (x+a, y+b) is visible from origin (i.e., gcd (x+a, y+b) = 1.)

By [2.1(iv)],

$$\gcd(m, n) = \gcd(m, n - m) \text{ for all } m, n \in Z,$$

we obtain

$$\gcd(x + a, y + b) = \gcd(x + a, (y + b) - (x + a)) = \gcd(x + a, y - x + b - a).$$

Hence,

$$\gcd(x + a, y + b) = 1 \Leftrightarrow \gcd(x + a, y - x + b - a) = 1.$$

Therefore, the translated point (x+a, y+b) is visible from origin iff gcd (x+a, y−x+b−a) = 1.

This completes the proof.

Remark. The above characterization shows that general translations preserve visibility only conditionally in sharp contrast with unimodular linear transformations, which preserve visibility unconditionally.

5 Conclusion

In this paper, we explored lattice-point visibility through both linear and nonlinear transformations, extending the classical role of co-primality. While unimodular linear maps preserve visibility, we identified new nonlinear families that also maintain this property. Our results highlight the critical and delicate role of divisibility, showing that even minor arithmetic changes can break visibility. This work lays the groundwork for a broader and deeper arithmetic theory of visibility beyond the linear setting.

Declaration of Competing Interest

the authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Conflict of Interest Statement

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