FUZZY IDEALS AND COMMUTATIVE FUZZY IDEALS OF BCH-ALGEBRA ACCORDING TO *T*-NORMS

Rasul Rasuli

Department of Mathematics, Payame Noor University (PNU)
P. O. Box 19395-4697, Tehran, Iran.

Sebar Ghaderi

Department of Mathematics, Payame Noor University (PNU)
P. O. Box 19395-4697, Tehran, Iran.

Abstract. In this paper, the concepts of fuzzy ideals and commutative fuzzy ideals of *BCH*-algebra according to *T*-norms are introduced and studied a characterization theorem of them. Relationship between them and ideals and commutative ideals of *BCH*-algebras are given. The intersection, fuzzy relation and cartesian product of families of them are established. We discussed the homomorphic image and preimage of them under homomorphisms of *BCH*-algebras.

Key words . Algebra and orders, theory of fuzzy sets, norms, products and intersections, homomorphisms.

I. INTRODUCTION

Some implicational logics have contributed to give rise to the notions of a few abstract algebras such as *BCK*-algebras and *BCI*-algebras. Consequently, another class of algebras known as the class of *BCH*-algebras has been introduced in [6, 7]. It has been shown [2] that the class of *BCK/BCI*-algebras is a proper subclass of *BCH*-algebras. Several aspects of this algebra have been studied in [1- 4, 25]. The concept of a fuzzy set, which was introduced by Zadeh in his definitive paper [27], was applied by many researchers to generalize some of the basic concepts of algebras. In [5], Hong et al. applied the concept to *BCH*-algebras and studied fuzzy dot subalgebras of *BCH*-algebras. In [28, 29], Zulfiqar introduced The concept of a commutative fuzzy ideal of *BCH*-algebras and other superior levels of fuzzyfication and investigated subimplicative (α, β)-fuzzy ideal of *BCH*-algebras. Wang [26] introduced the notions of fuzzy quasi-associative ideal and other fuzzy ideals of *BCH*-algebras. In 2004, Jiang and Chen defined homomorphic image of fuzzy ideals of *BCH*-algebras and investigated properties of them [8]. Moradian et al. [10] studied fuzzy ideals and fuzzy soft ideals of *BCH*-Algebras. In previous works [12-24], by using norms, the authors investigated some properties of fuzzy algebraic structures. In this paper, by applying *T*-norms, we deal with the algebraic structure of fuzzy ideals ($FIT(X)$) and commutative fuzzy ideals ($CFIT(X)$) of *BCH*-algebra and we discussed the algebraic properties of them. We give several characterizations of them and established their intersection, fuzzy relation and cartesian product. We characterize $FIT(X)$ and $CFIT(X)$ with ideals and commutative ideals of *BCH*-

algebra *X*, respectively and will show that any *CFIT(X)* will be *FIT(X)* but he converse of this assertion may not be true. Finally we prove that the homomorphic image and preimage of *FIT*(*X*) and *CFIT*(*X*) under homomorphisms of *BCH*-algebras will be *FIT*(*X*) and *CFIT*(*X*), respectively.

II. PRELIMINARIES

This section contains some basic definitions and preliminary results which will be needed in the sequal. For more details we refer readers to [2, 5, 6, 8, 9, 11, 13, 14, 16, 21, 23, 24, 26].

Definition 2.1. By a *BCH*-algebra, we mean an algebra (*X*, •,0) of type (2,0) satisfying the axioms:

(1) $x \bullet x = 0$ (2) $x \bullet y = 0$ and $y \bullet x = 0$ imply $x = y$ (3) $(x \bullet y) \bullet z = (x \bullet z) \bullet y$

For all $x, y, z \in X$.

In a *BCH*-algebra, we can define a partial ordering $"\leq"$ by $x \leq y$ if and only if $x \bullet y = 0$.

A *BCH*-algebra *X* is said to be non-negative if $0 \bullet x = 0$ for all $x \in X$.

Proposition 2.2. *In any BCH-algebra X, the following are true, for all* $x, y \in X$.

(1) $x \bullet (x \bullet y) \leq y$ (2) $0 \bullet (x \bullet y) = (0 \bullet x) \bullet (0 \bullet y)$

(3) $x \bullet 0 = x$ (4) $x \leq 0$ implies $x = 0$

Definition 2.3. A non-empty subset *I* of a *BCH*-algebra *X* is called an ideal of *X* if

(1) $0 \in I$,

(2) $x \bullet y \in I$ and $y \in I$ imply $x \in I$, for all $x, y \in X$.

Definition 2.4. A non-empty subset *I* of a *BCH*-algebra *X* is called a commutative ideal of *X*, denoted by $I \rhd X$, if it satisfies (1) and (2), where

(1) $o \in I$,

(2) $(x \bullet y) \bullet z \in I$ and $z \in I$ imply $x \bullet ((y \bullet (y \bullet x)) \bullet (0 \bullet (0 \bullet (x \bullet y)))) \in I$, for all $x, y, z \in X$.

Proposition 2.5. A commutative ideal of a BCH-algebra must be an ideal, but the converse does not hold.

Proposition 2.6. Let X BCH-algebra. Then $I \rhd X$, if and only if $x \bullet y \in I$ then for all $x, y \in X$, $x \bullet ((y \bullet (y \bullet x)) \bullet (0 \bullet (0 \bullet (x \bullet y)))) \in I$.

Definition 2.7. Define a mapping $f : (X, \bullet, o) \to (X, *, 0)$ of *BCH*-algebras a homomorphism if $f(x \bullet) = f(x) * f(y)$, for all $x, y \in X$.

Definition 2.8. A fuzzy subset of an arbitrary *X*, we mean a function $\mu : X \to [0,1]$ and $[0,1]^X$ for all fuzzy subsets of *X*. We call the set $\mu_s = \{x \in X : \mu(x) \geq s\}$ an upper level of μ for all $s \in [0,1]$.

Definition 2.9. Let $\varphi : X \to Y$ be a function such that $\mu : X \to [0,1]$ and $\nu : Y \to [0,1]$. We define $\varphi(\mu)(y) = \sup\{\mu(x) \mid x \in X, \varphi(x) = y\}$ and $\varphi^{-1}(\nu)(x) = \nu(\varphi(x))$ for all $x \in X$ and $y \in Y$.

Definition 2.10. Define a *t*-norm T as $T: [0,1]\times[0,1]\to[0,1]$ such that for all $x,y,z\in[0,1]$:

(1) $T(x,1)=x$ (neutral element) (2) $T(x,y)\le T(x,z)$ if $y\le z$ (monotonicity)

(3) $T(x,y)=T(y,x)$ (commutativity) (4) $T(x,T(y,z))=T(T(x,y),z)$ (associativity)

It is clear that if $x_1\ge x_2$ and $y_1\ge y_2$, then $T(x_1,y_2)\ge T(x_2,y_2)$.

Example 2.11. (1) Standard intersection *t*-norm $T_m(x,y)=\min\{x,y\}$.

(2) Bounded sum *t*-norm $T_b(x,y)=\max\{0,x+y-1\}$.

(3) algebraic product *t*-norm $T_p(x,y)=xy$.

(4) Drastic *t*-norm , $T_D(x,y)=\begin{cases} y & if\ x=1 \\ x & if\ y=1 \\ 0 & otherwise \end{cases}$.

(5) Nilpotent minimum *t*-norm, $T_{nM}(x,y)=\begin{cases} \min\{x,y\} & if\, x+y>1 \\ 0 & otherwise. \end{cases}$

(6) Hamacher product *T*-norm , $T_{H_0}(x,y)=\begin{cases} 0 & if\ \ x=y=0 \\ \dfrac{xy}{x+y+xy} & otherwise. \end{cases}$

The drastic *t*-norm is the pointwise smallest *t*-norm and the minimum is the pointwise largest *t*-norm: $T_D(x,y)\le T(x,y)\le T_{\min}(x,y)$ for all $x,y\in[0,1]$

We say that T be idempotent if for all $x\in[0,1]$ we have $T(x,x)=x$.

Definition 2.12. The functions $T_n:\prod_{i=1}[0,1]\to[0,1]$ are defined by

$$T_n(x_1,x_2,...,x_n)=T(x_i,T_{n-1}(x_1,x_2,...,x_{i-1},x_{i+1},...,x_n))$$

For all $1\le i\le n,$ where $n\le 2$ such that $T\ _2=T$ and $T_1=id$ (identity).

Using the induction on n, we have the following two lemmas.

Lemma 2.13. Let T be a t-norm. Then for every $x_i, y_i\in[0,1],$ where $1\le i\le n,$ and $n\ge 2,$ we have

$$T_n(T(x_1,y_1),T(x_2,y_2),...,T(x_n,y_n))=T(T_n(x_1,x_2,...,x_n),T_n(y_1,y_2,...,y_n)).$$

Lemma 2.14. Let T be a t-norm. Then for every $x_1,x_2,...,x_n\in[0,1],$ where $n\ge 2,$ we have

$$T_n(x_1,x_1,...,x_n)=T(...T(T(T(x_1,x_2),x_3),x_4),x_n)=T(x_1T(x_2,T(x_3,...T(x_3,...,T(x_{n-1},x_n)...))).$$

Definition 2.15. Let $\mu,v:X\to[0,1]$ define $\mu\cap v:X\to[0,1]$ as $(\mu\cap v)(x)=T(\mu(x),v(x))$

For all $x\in X$.

Definition 2.16. Let $\mu : X \to [0,1]$ and $v : Y \to [0,1]$ The Cartesian product of μ and v is denoted by $\mu \times v : X \times Y \to [0,1]$ is defined by $(\mu \times v)(x, y) = T(\mu(x), v(y))$ for all $(x, y) \in X \times Y$.

Lemma 2.17. Let T be a t-norm. Then for all $x , y ,w ,z \in [0,1]$:

$$T(T(x, y), T(w, z)) = T(T(x, w), T(y, z)),$$

III. FUZZY IDEALS AND COMMUTATIVE FUZZY IDEALS OF *BCH*-ALGEBRA ACCORDING TO *T*-NORMS

Definition 3.1. Define $\mu : X \to [0,1]$ as a fuzzy ideal of *BCH*-algebra X under *t*-norm *T* if it satisfies the following inequalities for all $x , y \in X$:

(1) $\mu(0) \geq \mu(x)$ (2) $\mu(x) \geq T(\mu(x \bullet y), \mu(y))$

Denote by *FIT (X)*, the set of all fuzzy ideals of *X* under *t*-norm *T*.

Example 3.2. Let $X = \{0, 2, 4, 6\}$ be a set as following Cayley table:

•	0	2	4	6
0	0	0	0	0
2	2	0	0	4
4	4	4	0	0
6	6	6	6	0

Then $(X, \bullet, 0)$ is a *BCH*-algebra. Define $\mu : X \to [0,1]$ as $\mu(x) = \begin{cases} 0.55 & if \ \ x = 0 \\ 0.35 & if \ \ x = 2 \\ 0.45 & if \ \ x = 4 \\ 0.25 & if \ \ x = 6 \end{cases}$

Let $T(a,b) = T_p(a,b) = ab$ for all $a, b \in [0,1]$ hence $\mu \in FIT(X)$.

Proposition 3.3. If $\mu \in FIT(X)$, then

(1) μ is order reversing, *i.e.*, $x \leq y$ implies $\mu(x) \geq \mu(y)$, for all $x, y \in X$.

(2) $\mu(x \bullet (x \bullet y)) \geq \mu(y)$, for all $x, y \in X$.

Proposition 3.4. Let $\mu \in FIT(X)$ of a non-negative BCH-algebra X. Then $\mu(x \bullet y) \geq \mu(x)$ for all $x, y \in X$.

Proposition 3.5. Let μ be a fuzzy set of non-negative BCH-algebra X. Then $\mu \in FIT(X)$ if and only if $x \bullet y \leq z$ implies $\mu(x) \geq T(\mu(y), \mu(z))$ for all $x, y, z \in X$.

Proof. Suppose $\mu \in FIT(X)$ and $x \bullet y \leq z$ then Proposition 3.3 gives us $\mu(x \bullet y) \geq \mu(z)$ and so $\mu(x) \geq T(\mu(x \bullet y), \mu(y)) \geq T(\mu(z), \mu(y)) = T(\mu(y), \mu(z))$.

Conversely, let $x \bullet y \leq z$ implies $\mu(x) \geq T(\mu(y), \mu(z))$. As Proposition 2.2 we have $x \bullet (x \bullet y) \leq y$ *so* $\mu(x) \geq T(\mu(x \bullet y), \mu(y))$ hence $\mu \in FIT(X)$.

Proposition 3.6. Let μ be a fuzzy set of non-negative BCH-algebra X. Then $\mu \in FIT(X)$ if and only if $x \bullet y \leq z$ implies $\mu(x) \geq T(\mu(y), \mu(z))$ for all $x, y, z \in X$.

Proof. If a *BCH*-algebra X is non-negative, then $0 \bullet x = 0$ then $(0 \bullet x) \bullet x = 0 \bullet x = 0$ hence $0 \bullet x \leq x$ for all $x \in X$. then $\mu(0) \geq T(\mu(x), \mu(x)) = \mu(x)$. Therefore Proposition is true by Proposition 3.5.

Proposition 3.7. Let T be idempotent. Then $\mu \in FIT(X)$ if and only if $\mu_s = \{x \in X : \mu(x) \geq s\} = \varnothing$ be an ideal of BCH-algebra X for every $s \in (0,1]$.

Proof. Let $\mu \in FIT(X)$ *and* $x, y \in X$. Then $\mu(0) \geq \mu(x) \geq s$ and then $0 \in \mu_s$. Also let $x \bullet y \in \mu_s$ and $y \in \mu_s$. Then $\mu(x) \geq T(\mu(x \bullet y), \mu(y)) \geq T(s,s) = s$ thus $x \in \mu_s$ will be an ideal of *BCI*-algebra *X* for every $s \in [0,1]$.

Conversely, let μ_s be either empty or an ideal of *BCH*-algebra *X* for every $s \in [0,1]$. Let $s = T(\mu(x \bullet y), \mu(y))$ *with* $x \bullet y \in \mu_s$ and $y \in \mu_s$. Then $x \in \mu_s$ thus

$$\mu(x) \geq s = T(\mu(x \bullet y), \mu(y))$$

so $\mu \in FIT(X)$.

Example 3.8. Let $X = \{0, 2, 4, 6, 8, 10\}$ be a set as following Cayley table:

$\bullet$	0	2	4	6	8	10
0	0	0	0	0	8	8
2	2	0	0	2	8	8
4	4	4	0	4	8	8
6	6	6	6	0	8	8
8	8	8	8	8	0	0
10	10	10	10	10	2	0

Then $(X, \bullet, 0)$ is a *BCH*-algebra. Define $\mu : X \to [0,1]$ as

$$\mu(x) = \begin{cases} 0.9 & if \ \ x = 0 \\ 0.7 & if \ \ x = 2 \\ 0.6 & if \ \ x = 4 \\ 0.5 & if \ \ x = 6 \\ 0.3 & if \ \ x = 8 \\ 0.1 & if \ \ x = 10 \end{cases}$$

Let $T(a,b) = T_p(a,b) = ab$ for all $a, b \in [0,1]$ hence $\mu \in FIT(X)$. Assume $s = 0.5$ which $\mu_{0.5} = \{x \in X : \mu(x) \geq 0.5\} = \{0, 2, 4, 6\}$ such that

$\bullet$	0	2	4	6
0	0	0	0	0
2	2	0	0	2
4	4	4	0	4
6	6	6	6	0

and so $\mu_{0.5}$ will be an ideal of *BCH*-algebra X.

Corollary 3.9. Let T be idempotent. Then $\mu \in FIT(X)$ if and only if $A_{x_0} = \{x \in X : \mu(x) \geq \mu(x_0)\} \neq \varnothing$ be an ideal of BCH-algebra X for every $x_0 \in X$.

Corollary 3.10. Let T be idempotent. Then $\mu \in FIT(X)$ if and only if $A = \{x \in X : \mu(x) = \mu(0) \neq \varnothing$ be an ideal of BCH-algebra X.

Definition 3.11. Let $\mu : X \to [1.0]$ *bea fuzzy subset of X. Define* $\mu^{\lambda} : X \to [0,1]$ *as* $\mu^{\lambda}(x) = T(\lambda, \mu(x))$ for all $\lambda \in [0,1]$ and $x \in X$. Also we can define $\mu^{\mu(a)} : X \to [0,1]$ which $\mu^{\mu(a)}(x) = T(\mu(a), \mu(x))$ for all $a, x \in X$.

Proposition 3.12. *Let* $\mu \in FIT(x)$ *and T be idempotent. Then* $\mu^{\lambda} \in FIT(X)$ *for all* $\lambda \in [0,1]$.

Thus $\mu^{\lambda}(x) \geq T(\mu^{\lambda}(x \bullet y), \mu^{\lambda}(y))$. Then $\mu^{\lambda} \in FIT(X)$.

Example 3.13. Let $X = \{0, 1000, 2000, 3000\}$ *be a set as following Cayley table:*

$\bullet$	0	1000	2000	3000
0	0	1000	2000	3000
1000	1000	0	3000	2000
2000	2000	3000	0	1000
3000	3000	2000	1000	0

Then $(X, \bullet, 0)$ is a *BCH*-algebra. Define $\mu : X \to [0,1]$ as

$$\mu(x) = \begin{cases} 0.5 & if \ \ x = 0 \\ 0.3 & if \ \ x = 1000 \\ 0.4 & if \ \ x = 2000 \\ 0.2 & if \ \ x = 3000 \end{cases}$$

and $\mu^{0.8} : X \to [0,1]$ as $\mu^{0.8}(x) = T(0.8, \mu(x))$ for $0.8 \in [0,1]$. Let $T(a,b) = T_p(a,b) = ab$ *for all* $a, b \in [0,1]$ hence $\mu \in FIT(X)$ and $\mu^{\lambda} \in FIT(X)$.

Proposition 3.14. Let $\mu \in FIT(X)$ and T be idempotent. Then $\mu^{\mu(a)} \in FIT(X)$ for all $a \in [0,1]$.

Proposition 3.15. Let $\mu \in FIT(X)$ then $\mu^{f}(x) = f(\mu(x)) \in FIT(X)$ for all $x \in X$ and increasing functions $f : [0,1] \to [0,1]$. Then $\mu^{f} \in FIT(X)$.

Proposition3.16. Let $\mu \in FIT(X)$ of a non-negative BCH-algebra X. Then

$v(x) = \inf\{\mu(x \bullet y) : y \in X\} \in FIT(X)$ for all $x \in X$.

Proposition 3.17. Let $\mu \in FIT(X)$ and $v \in FIT(X)$. Then $\mu \cap v \in FIT(X)$.

Example 3.18. Let $X = \{0, 1, 2, 3, 4\}$ be a set as following Cayley table:

•	0	1	2	3	4
0	0	0	0	3	4
1	1	0	1	4	3
2	2	2	0	3	3
3	3	3	3	0	0
4	4	3	4	1	0

Then $(X,\bullet,0)$ is a *BCH*-algebra. Define fuzzy subsets $\mu,v:(X,\bullet,0)\to[0,1]$ as

$$\mu(x)=\begin{cases}0.5 & if \;\; x=0\\ 0.3 & if \;\; x=1,2\\ 0.1 & if \;\; x=3,4\end{cases}\quad , \quad v(x)=\begin{cases}0.7 & if \;\; x=0,3\\ 0.4 & if \;\; x=1,4\\ 0.2 & if \;\; x=2\end{cases}$$

let $T(a,b)=T_p(a,b)=ab$ for all $a,b\in[0,1]$ then $\mu,v\in FIT(X)$. Also $\mu\cap v:X\to[0,1]$ will be as

$$(\mu\cap v)(x)=T_p(\mu(x),(v(x))=\mu(x)v(x)=\begin{cases}0.35 & if \;\; x=0\\ 0.12 & if \;\; x=1\\ 0.06 & if \;\; x=2\\ 0.07 & if \;\; x=3\\ 0.04 & if \;\; x=4\end{cases}$$

Thus $\mu\cap v\in FIT(X)$.

Corollary 3.19. Let $\mu_i\subseteq FIT(X)$ for $i=1,2,3,4,...,n$. Then $\cap_{i=1,2,3,...,n}\mu_i\in FIT(X)$.

Proposition 3.20. Let $\mu\in FIT(X)$ and $v\in FIT(Y)$. Then $\mu\times v\in FIT(X\times Y)$.

Example 3.21. Let $X=\{1,2,3,5\}$ be a set as following Cayley table:

•	0	1	3	5
0	0	0	0	0
1	1	0	5	5
3	3	0	0	3
5	4	0	0	0

Then $(X,\bullet,0)$ is a *BCH*-algebra. Also let $Y=\{0,-1,-3,-5\}$ be a set with

•	0	-1	-3	-5
0	0	0	-5	-3
-1	-1	0	-5	-3
-3	-3	-3	0	-5
-5	-5	-5	-3	0

Define fuzzy subsets $\mu:(X,\bullet,0)\to[0,1]$ as

$$\mu(x)=\begin{cases}0.5 & if \;\; x=0\\ 0.4 & if \;\; x=1\\ 0.3 & if \;\; x=3\\ 0.2 & if \;\; x=5\end{cases}$$

and $v:(Y,\bullet,0)\rightarrow[0,1]$ which

$$v(y)=\begin{cases}0.6 & if \;\; y=0\\ 0.2 & if \;\; y=-1\\ 0.1 & if \;\; y=-3\\ 0.4 & if \;\; y=-5\end{cases}$$

let $T(a,b)=T_p(a,b)=ab$ for all $a,b\in[0,1]$ then $\mu\in FIT(X)$ and $v\in FIT(Y).$ We get that $X\times Y=\{(0,0),(0,-1),(0,-3),(0,-5),(1,0),(1,-1),(1,-3),(1,-5),(3,0),(3,-1),(3,-3),(3,-5),(5,0),$ $(5,-1),(5,-3),(5,-5)].$ Define $\mu\times v:X\times Y\rightarrow[0,1]$ as

$$(\mu\times v)(x,y)=T_p(\mu(x),(v(y))=\mu(x)v(y)=\begin{cases}0.3 & if \;\; (x,y)=(0,0)\\ 0.1 & if \;\; (x,y)=(0,-1)\\ 0.05 & if \;\; (x,y)=(0,-3)\\ 0.2 & if \;\; (x,y)=(0,-5)\\ 0.24 & if \;\; (x,y)=(1,0)\\ 0.08 & if \;\; (x,y)=(1,-1),(5,-5)\\ 0.04 & if \;\; (x,y)=(1,-3),(5,-1)\\ 0.16 & if \;\; (x,y)=(1,-5)\\ 0.18 & if \;\; (x,y)=(3,0)\\ 0.06 & if \;\; (x,y)=(3,-1)\\ 0,03 & if \;\; (x,y)=(3,-3)\\ 0.12 & if \;\; (x,y)=(3,-5),(5,0)\\ 0.02 & if \;\; (x,y)=(5,-3)\end{cases}$$

Hence $\mu\times v\in FIT(X\times Y).$

Corollary 3.22. Let $\mu_i\in FIT(X_i)$ where $1\leq i\leq n,$ then

$$\mu=\prod_{i=1}^{n}\mu_i\in FIT(\prod_{i=1}^{n}X_i=X)$$

with

$$\mu(x)=\prod_{i=1}^{n}\mu_i(x_1,x_2,...,x_n)=T_n(\mu_1(x_1,\mu_2(x_2,...,\mu_n(x_n)))$$

for all $x=(x_1,x_2,...,x_n)\in X$.

Proposition 3.23. Let μ and v be fuzzy subsets of a BCH-algebra X. if $\mu\times v\in FIT(X\times X)$, then

(1) $\mu(0)\geq\mu(x)\, or\, v(0)\geq v(x)\, for\ all\ x\in X$ (2) $v(0)\geq\mu(x)\, or\, v(0)\geq v(x)\, for\ all\ x\in X$

(3) $\mu(0)\geq\mu(x)\, or\ \mu(0)\geq v(x)\, for\ all\ x\in X$ (4) $\mu\in FIT(X)\, or\, v\in FIT(X)$.

Proof. We prove only (4). Let $x,y\in X$ and from (1) we get that $\mu(0)\geq\mu(x)$. As (2) we give $(\mu\times v)(x,0)=T(\mu(x),v(0))=\mu(x)$ and as $\mu\times v\in FIT(X\times X)$ we will have

$$\begin{aligned}(\mu\times v)(x,0)&\geq T((\mu\times v)(x,0)\bullet(y,0)),(\mu\times v)(y,0))\\&=T((\mu\times v)(x\bullet y,0\bullet 0),(\mu\times v)(y,0))\\&=T((\mu\times v)(x\bullet y,0),(\mu\times v)(y,0))\\&=T(\mu(x\bullet y),\mu(y))\end{aligned}$$

thus $(\mu\times v)(x,0)\geq T(\mu(x\bullet y),(\mu(y))$. *Therefore* $\mu\in FIT(X)$. The proof of $v\in FIT(X)$ is similar.

Proposition 3.24. Let X be a non-negative BCH-algebra and $\mu\in FIT(X\times X)$. Then

(1) $\mu(x\bullet y,z\bullet t)\geq\mu(x,z)$ for all $x,y,z,t\in X$.

(2) $\mu(x,0)\geq\mu(x,y)$ and $\mu(0,y)\geq\mu(x,y)$ for all $x,y\in X$.

Proof.

(1) As $(x\bullet y,z\bullet t)\bullet(x,z)=((x\bullet y)\bullet x,(z\bullet t)\bullet z)=((x\bullet x)\bullet y),(z\bullet z)\bullet t)=(0\bullet y,0\bullet t)=(0,0)$ hence $(x\bullet y,z\bullet t)\leq(x,z)$. Now Proposition 3.3 gives us that $\mu(x\bullet y,z\bullet t)\geq\mu(x,z)$.

(2) Since $(x,0)\bullet(x,y)=(x\bullet x,0\bullet y)=(0,0)$ and $(0,y)\bullet(x,y)=(0\bullet x,y\bullet y)=(0,0)$ so $(x,0)\leq(x,y)$ and $(0,y)\leq(x,y)$ then $\mu(x,0)\geq\mu(x,y)$ and $\mu(0,y)\geq\mu(x,y)$ for all $x,y\in X$.

Proposition 3.25. Let X be a non-negative BCH-algebra and $\mu\in FIT(X\times X)$. Let $\mu^a:X\to[0,1]$ which $\mu^a(x)=\mu(x,a)$ for all $a,x\in X$. Then $\mu^a\in FIT(X)$.

Corollary 3.26. Let X be a non-negative BCH-algebra and $\mu\in FIT(X\times X)$. Let $\mu^a:X\to[0,1]$ which $\mu^a(x)=\mu(a,x)$ for all $a,x\in X$. Then $\mu^a\in FIT(X)$.

Definition 3.27. Let $\mu,v:X\times X\to[0,1]$ *be two fuzzy relations on a set X. Then* $\mu\circ v:X\times X\to[0,1]$ is a fuzzy relation on a set *X*, where it defined as

$$(\mu\circ v)(x,y)=\sup\{T(\mu(x,z),v(z,y)):z\in X\}$$

For all $x,y,z\in X$.

Proposition 3.28. If $\mu\in FIT(X)$ and $\varphi:(X,\bullet,0)\to(Y,*,0)$ be a homomorphism of BCH-algebras, then $\varphi(\mu)\in FIT(Y)$.

Proposition 3.29. If $v \in FIT(Y)$ and $\varphi:(X,\bullet,0) \to (Y,*,0)$ be a homomorphism of BCH-algebras, then $\varphi^{-1}(v) \in FIT(X)$.

Example 3.30. Let $X = \{1,2,3,4,5\}$ be a set as following Cayley table:

$\bullet$	0	1	2	3	4	5
0	0	0	0	0	4	4
1	1	0	0	1	4	4
2	2	2	0	2	5	4
3	3	3	3	0	4	4
4	4	4	4	4	0	0
5	5	5	4	5	2	0

Then $(X,\bullet,0)$ is a *BCH*-algebra. Also let $Y = \{0,-1,-2,-3\}$ be a set with

$\bullet$	0	-1	-2	-3
0	0	0	0	0
-1	-1	0	0	-1
-2	-2	-2	0	-2
-3	-3	-3	-3	0

Define fuzzy subsets $\mu:(X,\bullet,0) \to [0,1]$ by

$$\mu(x) = \begin{cases} 0.6 & if \quad x = 0 \\ 0.5 & if \quad x = 1 \\ 0.4 & if \quad x = 2 \\ 0.2 & if \quad x = 3 \\ 0.1 & if \quad x = 4 \\ 0.3 & if \quad x = 5 \end{cases}$$

and $v:(Y,\bullet,0) \to [0,1]$ which

$$v(y) = \begin{cases} 0.5 & if \quad y = 0 \\ 0.4 & if \quad y = -1 \\ 0.3 & if \quad y = -2 \\ 0.2 & if \quad y = -3 \end{cases}$$

let $T(a,b) = T_m(a,b) = \min\{a,b\}$ for all $a,b \in [0,1]$ then $\mu \in FIT(X)$ and $v \in FIT(Y)$.

Define $\varphi: X \to Y$ as

$$\varphi(x) = \begin{cases} 0 & if \quad x = 0 \\ -1 & if \quad x = 1,4 \\ -2 & if \quad x = 2,3 \\ -3 & if \quad x = 5 \end{cases}$$

then

$$\varphi(\mu)(y)=\begin{cases}0.6 & if \quad y=0\\ 0.5 & if \quad y=-1\\ 0.4 & if \quad y=-2\\ 0.3 & if \quad y=-3\end{cases}$$

therefore $\varphi(\mu)\in FIT(Y)$. Also $\varphi^{-1}:Y\to X$ will be as

$$\varphi^{-1}(v)(x)=v(\varphi(x))=\begin{cases}0.5 & if \quad x=0\\ 0.4 & if \quad x=1,4\\ 0.3 & if \quad x=2,3\\ 0.2 & if \quad x=5\end{cases}$$

then $\varphi^{-1}(v)\in FIT(X)$.

Definition 3.31. A fuzzy set μ of a *BCH*-algebra *X* is called a commutative fuzzy ideal of *X* according to *t*-norm *T* if it satisfies (1) and (2), where

(1) $\mu(0)\geq\mu(x)$,

(2) $\mu(x\bullet((y\bullet(y\bullet x))\bullet(0\bullet(0\bullet(x\bullet y)))))\geq T(\mu((x\bullet y)\bullet z),\mu(z))$,

for all $x,y,z\in X$. Denote by *CFIT(X)*, the set of all commutative fuzzy ideals of *X* under *t*-norm *T*.

Proposition 3.32. Assume T be idempotent. Then $\mu\in CFIT(X)$ if and if $\mu_s=\{x\in X:\mu(x)\geq s\}\varnothing$ be a commutative ideal of BCH-algebra X for every $s\in(0,1]$.

Corollary 3.33. Let T be idempotent. Then $\mu\in CFIT(X)$ if and only if $A_{x_0}=\{x\in X:\mu(x)\geq\mu(x_0)\}\neq\varnothing$ be a commutative ideal of BCH-algebra X for every $x_0\in X$.

Corollary 3.34. Let T be idempotent. Then $\mu\in CFIT(X)$ if and only if $A=\{x\in X:\mu(x)=\mu(0)\}\neq\varnothing$ be a commutative ideal of BCH-algebra X.

Proposition 3.35. Let $\mu\in CFIT(X)$ and $v\in CFIT(X)$. Then $\mu\cap v\in CFIT(X)$.

Example 3.36. Let $X=\{0,-10,-20,-30\}$ be a set as following Cayley table:

•	0	-10	-20	-30
0	0	0	0	0
-10	-10	0	0	0
-20	-20	-20	0	0
-30	-30	-30	-30	0

Then $(X,\bullet,0)$ is a *BCH*-algebra.

Define fuzzy subsets $\mu,v:(X,\bullet,0)\to[0,1]$ as

$$\mu(x)=\begin{cases}0.9 & if \;\; x=0\\ 0.2 & if \;\; x=-10\\ 0.1 & if \;\; x=-20\\ 0.4 & if \;\; x=-30\end{cases}$$

and

$$v(x)=\begin{cases}0.7 & if \;\; x=0\\ 0.5 & if \;\; x=-10,-30\\ 0.8 & if \;\; x=-20\end{cases}$$

let $T(a,b)=T_p(a,b)=ab$ for all $a,b\in[0,1]$ then $\mu,v\in CFIT(X)$.

Also $\mu\cap v: X\to[0.1]$ will be as

$$(\mu\cap v)(x)=T_P(\mu(x),v(x))=\mu(x)v(x)=\begin{cases}0.63 & if \;\; x=0\\ 0.1 & if \;\; x=-10\\ 0.08 & if \;\; x=-20\\ 0.2 & if \;\; x=-30\end{cases}$$

thus $\mu\cap v\in CFIT(X)$.

Corollary 3.37. Let $\mu_i\subseteq CFIT(X)$ for $i=1,2,3,4,...,n.$ Then $\cap_{i=1,2,3,...,n}\mu_i\in CFIT(X)$.

Proposition 3.38. Let $\mu\in CFIT(X)$ and $v\in CFIT(y)$. Then $\mu\times v\in CFIT(X\times Y)$.

Example 3.39. Let $X=\{0,10,20,30\}$ be a set as following Cayley table:

●	0	10	20	30
0	0	0	0	0
10	10	0	0	0
20	20	10	0	10
30	30	30	30	0

Then $(X,\bullet,0)$ is a BCH-algebra. Also let $Y=\{0,-1,-2,-3\}$ be a set with

●	0	-1	-2	-3
0	0	0	0	0
-1	-1	0	0	0
-2	-2	-1	0	-1
-3	-3	-1	-1	0

Define fuzzy subsets $\mu:(X,\bullet,0)\to[0,1]$ as

$$\mu(x)=\begin{cases}0.6 & if \;\; x=0\\ 0.1 & if \;\; x=10\\ 0.4 & if \;\; x=20\\ 0.3 & if \;\; x=30\end{cases}$$

and $v:(Y,\bullet,0)\to[0,1]$ which

$$v(y)=\begin{cases}0.8 & if \quad y=0\\ 0.5 & if \quad y=-1\\ 0.3 & if \quad y=-2\\ 0.1 & if \quad y=-3\end{cases}$$

let $T(a,b)=T_p(a,b)=ab$ for all $a,b\in[0,1]$ then $\mu\in CFIT(X)$ and $v\in CFIT(Y)$. We get that $X\times Y=(0,0),(0,-1),(0,-2),(0,-3),(10,0),(10,-1),(10,-2),(10,-3),(20,0),(20,-1),(20,-2),(20,-3),$ $(30,0),(30,-1),(30,-2),(30,-3).$ Define $\mu\times v: X\times Y\to[0,1]$ as

$$(\mu\times v)(x,y)=T_p(\mu(x),(v(y))=\mu(x)v(y)=\begin{cases}0.48 & if \quad (x,y)=(0,0)\\ 0.3 & if \quad (x,y)=(0,-1)\\ 0.18 & if \quad (x,y)=(0,-2)\\ 0.06 & if \quad (x,y)=(0,-3)\\ 0.08 & if \quad (x,y)=(10,0)\\ 0.05 & if \quad (x,y)=(10,-1)\\ 0.03 & if \quad (x,y)=(10,-2),(30,-3)\\ 0.01 & if \quad (x,y)=(10,-3)\\ 0.32 & if \quad (x,y)=(20,0)\\ 0.2 & if \quad (x,y)=(20,-1)\\ 0.12 & if \quad (x,y)=(20,-2)\\ 0.04 & if \quad (x,y)=(20,-3)\\ 0.24 & if \quad (x,y)=(30,0)\\ 0.15 & if \quad (x,y)=(30,-1)\\ 0.09 & if \quad (x,y)=(30,-2)\end{cases}$$

hence $\mu\times v\in CFIT(X\times Y).$

Corollary 3.40. Let $\mu_i\in CFIT(X_i)$ where $1\le i\le n,$ then

$$\mu=\prod_{i=1}^{n}\mu_i\in CFIT(\prod_{i=1}^{n}X_i=X) \quad \text{with} \quad \mu(x)=\prod_{i=1}^{n}\mu_i(x_1,x_2,...,x_n)=T_n(\mu_1(x_1,\mu_2(x_2,...,\mu_n(x_n)))$$

for all $x=(x_1,x_2,...,x_n)\in X.$

Proposition 3.41. If $\mu\in CFIT(X)$ and $\varphi:(X,\bullet,0)\to(Y,*,0)$ be a homomorphism of BCH-algebras, then $\varphi(\mu)\in CFIT(Y)$.

Proposition 3.42. If $v\in CFIT(Y)$ and $\varphi:(X,\bullet,0)\to(Y,*,0)$ be a homomorphism of BCH-algebras then $\varphi^{-1}(v)\in CFIT(X).$

Proposition 3.43. Any CFIT(X) will be FIT(X).

Proof. Let $\mu \in CFIT(X)$ *and let* $x, z \in X$. *Then*

$$\begin{aligned}\mu(x) &= \mu(x \bullet 0)\\ &= \mu(x \bullet ((0 \bullet (0 \bullet x)) \bullet (0 \bullet (0 \bullet x))))\\ &= \mu(x \bullet ((0 \bullet (0 \bullet x)) \bullet (0 \bullet (0 \bullet (x \bullet 0)))))\\ &\geq T(\mu((x \bullet 0) \bullet z), \mu(z))\\ &= T(\mu(x \bullet z), \mu(z))\end{aligned}$$

thus $\mu(x) \geq T(\mu(x \bullet z), \mu(z))$. Consequently $\mu \in FIT(X)$.

Remark 3.44. The converse of the above proposition may not be true as shown in the following example.

Example 3.45. Let $X = \{0, 100, 200, 300, 400\}$ be a set as following Cayley table:

$\bullet$	0	100	200	300	400
0	0	0	0	0	0
100	100	0	100	0	0
200	200	200	0	0	0
300	300	300	300	0	0
400	400	300	400	300	0

$(X, \bullet, 0)$ is a *BCH*-algebra. Let $t_0, t_1, t_2 \in [0,1]$ be such that $t_0 > t_1 > t_2$. Define $\mu : X \to [0,1]$ as

$$\mu(x) = \begin{cases} t_0 & if \;\; x = 0 \\ t_1 & if \;\; x = 100 \\ t_2 & if \;\; x = 200, 300, 400. \end{cases}$$

By simple calculations show that $\mu \in FIT(X)$ but $\mu \notin CFIT(X)$. Put x = 200, y = 300, z = 0 in

$$\mu(x \bullet ((y \bullet (y \bullet x)) \bullet (0 \bullet (0 \bullet (x \bullet y))))) \geq T(\mu((x) \bullet z), \mu(z))$$

we get that

$$\mu(200 \bullet ((300 \bullet 200)) \bullet (0 \bullet (0 \bullet (200 \bullet 300))))) \geq T(\mu((200 \bullet 300) \bullet), \mu(0))$$

Thus

$$\mu(200 \bullet ((300 \bullet 300) \bullet (0 \bullet (0 \bullet 0)))) \geq T(\mu(0 \bullet 0), \mu(0))$$

hence

$$\mu(200 \bullet (0 \bullet (0 \bullet 0))) \geq T(\mu(0), \mu(0))$$

and $\mu(200 \bullet (0 \bullet 0)) \geq T(\mu(0), \mu(0)) = \mu(0)$ so $\mu(200 \bullet 0)) \geq t_0$ thus $\mu(200) \geq t_0$ then $t_2 \geq t_0$ and this a contradiction with $t_0 > t_1 > t_2$. Hence $\mu \notin CFIT(X)$.

Proposition 3.46 . Assume $\mu \in FIT(X)$. Then $\mu \in CFIT(X)$ if and only if

$$\mu(x \bullet ((y \bullet (y \bullet x)) \bullet (0 \bullet (0 \bullet (x \bullet y))))) \geq \mu(x \bullet y) (*)$$

For all $x, y \in X$.

IV. CONCLUSION AND OPER PROBLEM

In this study, as using *T*-norms, we defined fuzzy ideals and commutative fuzzy ideals of *BCH*-algebra and investigated algebraic structure of them. Now one can define and investigated fuzzy ideals and commutative fuzzy ideals of generalized *BCH*-algebras according to *T*-norms as we did for *BCH*-algebras and this can be an open problem for future works.

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