Generalized Monotone Method for Matrix Differential Equations with Nonlinear Boundary Condition

S. N. R. G. Bharat Iragavarapu[1], Ch.Satyanarayana[2]

[1] Department of Mathematics,

Gayatri Vidya Parishad College of Engineering (Autonomous), Visakhapatnam, AP, India

[2] Department of Mathematics, B. V Raju College (Autonomous), Bhimavaram, AP, India

Abstract

In this paper the concept of coupled quasisolutions is studied for a nonlinear boundary value problem (BVP) of a Matrix Differential Equations (MDEs). Using the monotone iterative technique, we construct the coupled quasisolutions of a nonlinear BVP. We show that these iterates converge uniformly and monotonically to minimal and maximal solutions and are really coupled quasisolutions of nonlinear boundary value problem of matrix differential equations.

Keywords: Matrix differential equations, coupled lower and upper solutions, existence of a solution, uniqueness, monotone iterative technique, maximal and minimal solutions, coupled quasisolutions.

Introduction

Matrix differential equation is an excellent tool to solve a system of first order differential equations. The solution of a bunch of first order differential equations is much easier to find the solution of the equivalent higher order differential equations. In [1], J.Vasundhara Devi and et.al., utilized the concepts in [2] including matrix linear space. Observing that the study of MDEs falls into the realm of differential equations in abstract spaces, the author and et.al. planned to study MDEs. Monotone iterative technique [3, 4, 5] using method of upper and lower solutions is an effective tool and an important technique that provides existence and uniqueness of solutions for IVPs. In [6] the authors developed the MIT for PBVP of MDE. It is quite obvious that the study of boundary value problems (BVPs) for the considered equations is much more complicated than those of the corresponding PBVPs.

To determine the existence and uniqueness of solutions of the corresponding BVP, Pandit et al. [7] successfully applied the MIT for the PBVP. In [8, 9], MIT has been developed for nonlinear boundary value problems. Using the ideas in Pandit's paper the quasisolutions of IVP of MDE developed in [10]. In this paper, using the approach in [11] first we introduce the notion of coupled quasisolutions and consider the nonlinear BVP of matrix differential equation and obtain its solution through a sequence of iterates developed for the corresponding PBVP.

Preliminaries

In this section we give some definitions and state a few results pertaining to MDEs which are useful in proving the main result. Consider the MDE of the type

$$E' = F_1(t,E) + F_2(t,E) \tag{1}$$

with the nonlinear boundary condition

$$g\left(E(0), E(T)\right) = 0, \tag{2}$$

where $F_1, F_2 \in C\left[I \times R^{n\times n}, R^{n\times n}\right]$, $E \in C^1\left[I, R^{n\times n}\right]$.

Definition: The matrix functions $V_0, W_0 \in C^1\left[I, R^{n\times n}\right]$ are said to be lower and upper solutions of (1) if

$$V_0' \le F_1(t,V) + F_2(t,V), \quad t \in (0,T]$$

$$W_0' \ge F_1(t,W) + F_2(t,W), \quad t \in (0,T]$$

Now we start with the definitions of coupled lower and upper solutions of (1)-(2).

Definition: Let $V_0, W_0 \in C^1\left[I, R^{n\times n}\right]$. Then V_0, W_0 are said to be

(a) natural lower and upper solutions of (1) if

$$\left.\begin{aligned} &V_0' \le F_1(t,V_0) + F_2(t,V_0),\ g(V_0(0),V_0(T)) \le 0,\\ &W_0' \ge F_1(t,W_0) + F_2(t,W_0), g(W_0(0),W_0(T)) \ge 0,\ t \in I; \end{aligned}\right\} \tag{3}$$

(b) coupled lower and upper solutions of Type I of (1) if

$$\left.\begin{array}{l} V_0' \le F_1(t,V_0)+F_2(t,W_0),\ g(V_0(0),W_0(T)) \le 0, \\ W_0' \ge F_1(t,W_0)+F_2(t,V_0), g(W_0(0),V_0(T)) \ge 0\ t\in I; \end{array}\right\} \tag{4}$$

(c) coupled lower and upper solutions of Type II of (1) if

$$\left.\begin{array}{l} V_0' \le F_1(t,W_0)+F_2(t,V_0),\ g(W_0(0),V_0(T)) \le 0, \\ W_0' \ge F_1(t,V_0)+F_2(t,W_0),\ g(V_0(0),W_0(T)) \ge 0,\ t\in I; \end{array}\right\} \tag{5}$$

(d) coupled lower and upper solutions of Type III of (1) if

$$\left.\begin{array}{l} V_0' \le F_1(t,W_0)+F_2(t,W_0),\ g(W_0(0),W_0(T)) \le 0, \\ W_0' \ge F_1(t,V_0)+F_2(t,V_0), g(V_0(0),V_0(T)) \ge 0,\ t\in I. \end{array}\right\} \tag{6}$$

We observe that whenever we have $V(t)\le W(t), t\in I$, if $F_1(t,E)$ is nondecreasing in E for each $t\in I$ and $F_2(t,E)$ is nonincreasing in E for each $t\in I$, the lower and upper solutions defined by (3) and (6) reduce to (4) and (5) and consequently, it is sufficient to investigate the cases (4) and (5).

Definition: Let $V_0(t)$ and $W_0(t)$ be any two matrix sector functions with $W_0 \ge V_0$ then we denote the (functional interval) sector as $\langle V_0, W_0\rangle$ and is defined by

$$\langle V_0, W_0\rangle = \left\{ U\in C^1\left[J,R^{n\times n}\right] : V_0(t)\le U(t)\le W_0(t), t\in(0,T]\right\}.$$

Definition: A pair of functions ρ and r are in $C^1\left[I,R^{n\times n}\right]$ called coupled quasisolutions of the nonlinear BVP (1)-(2) if they are solutions of matrix differential equation (1) as well as satisfy the relations

$$V_0 \le \rho \le r \le W_0 \tag{7}$$

$$g\left(r(0),\rho(T)\right) = 0 = g\left(\rho(0), r(T)\right) \tag{8}$$

where $V_0(t)$ and $W_0(t)$ are coupled lower and upper solutions of the nonlinear BVP (1)-(2).

Further, we suppose that

$$\left.\begin{array}{ll} g(x,.) & \text{is nonincreasing for all } x\in C^1\left[I,R^{n\times n}\right], \\ g(.,y) & \text{is nondecreasing for all } y\in C^1\left[I,R^{n\times n}\right], \end{array}\right\} \tag{9}$$

Lemma: Let I=[0,T], $F_1, F_2 \in C\left[I, R^{n\times n}\right]$, $V_0, W_0 \in C^1\left[I, R^{n\times n}\right]$ are coupled lower and upper solutions of (1) and F_1, F_2 are quasi monotone nondecreasing and nonincreasing in E for each t. Further, suppose that F_1, F_2 satisfies the following Lipschitz condition

$$F_1(t, E_1) - F_2(t, E_2) \le L_1(E_1 - E_2),\ L_1 \in R^{n\times n}$$
$$F_2(t, E_1) - F_2(t, E_2) \ge L_2(E_1 - E_2),\ L_2 \in R^{n\times n}$$

Then $V_0 \le W_0$ implies that $V(t) \le W(t),\ t \in I.$

Corollary-1: Let $P \in C^1\left[I, R^{n\times n}\right]$ such that $P' \le MP(t)$ and $P(0) \le 0$ where $M \in R^{n\times n}$ then $P(t) \le 0$ for $0 \le t \le T$ and $M > 0.$

The result of Corollary-1 is true even if $M = 0,$ and is given below

Corollary-2: Let $P'(t) \le 0$ on [0,T]. If $P(0) \le 0$ then $P(t) \le 0,\ t \in I$

Generalized Monotone Method

In this section, based on the concepts defined above, we now develop the generalized monotone method for the nonlinear BVP (1)-(2) and obtain a minimal and maximal solutions for the PBVP of MDES.

In the following theorem, we use lower and upper solutions and obtain monotone sequences which converge uniformly and monotonically to minimal and maximal solutions of PBVP and we show that these solutions are coupled quasisolutions of nonlinear BVP defined by (1)-(2).

Theorem-1: Suppose that

(A1). $V_0, W_0 \in C^1\left[I, R^{n\times n}\right]$ are coupled lower and upper solutions of Type I for nonlinear BVP (1)-(2) with $V_0(t) \le W_0(t)$ on I,

(A2). the function $g(u,v) \in C\left[R^2, R\right]$ satisfies the condition (9),

(A3). $F_1, F_2 \in C\left[R^{n\times n}, R^{n\times n}\right]$ and $F_1(t, E_1)$ is non-decreasing in E_1 and $F_2(t, E_2)$ is nonincreasing in E_2 for each $t \in I.$

(A4). $F_1(t,E_1) - F_2(t,E_2) \le L_1(E_1 - E_2),\quad E_1 \ge E_2,\ L_1 \ge 0,$

(A5). $F_1(t,E_1) - F_2(t,E_2) \le L_1(E_1 - E_2),\ E_1 \ge E_2,\ L_2 \ge 0,$

Then the solution of nonlinear BVP (1)-(2) exists and is unique in the sector $\langle V_0, W_0 \rangle$.

Proof:

To complete the proof first we find minimal solution ρ and maximal solution r of the of the following IVP for nonlinear matrix differential equation

$$E' = F_1(t,E) + F_2(t,E) \tag{10}$$

with the initial condition

$$E(0) = E_0, \tag{11}$$

where $V_0 \le 0 \le W_0$. Further we show that minimal solution ρ and maximal solution r are truly coupled quasisolutions of nonlinear BVP (1)-(2). Secondly we show that it satisfy the nonlinear boundary condition (2). In order to develop and apply method of lower and upper solutions and its associated monotone iterative scheme to prove the existence of solution of IVP (10)-(11). We first define lower and upper solutions as follows.

Definition: Let $V_0, W_0 \in C^1\left[I, R^{n\times n}\right]$ be the coupled lower and upper solutions of Type I of (1) if

$$\left.\begin{aligned} &V_0' \le F_1(t,V_0) + F_2(t,W_0),\ V_0(0) \le 0,\\ &W_0' \ge F_1(t,W_0) + F_2(t,V_0),\ \ W_0(0) \ge 0\ \ t \in I; \end{aligned}\right\} \tag{12}$$

The following IVPs are the basis for the monotone iterative scheme given by

$$V^1_{n+1} = F_1(t,V_n) + F_2(t,W_n) \tag{13}$$

$$V_{n+1}(0) = 0, \tag{14}$$

$$W^1_{n+1} = F_1(t,W_n) + F_2(t,V_n) \tag{15}$$

$$W_{n+1}(0) = 0. \tag{16}$$

We know that corresponding to each of the linear IVPs (13)-(14) and (15)-(16) the solutions V_n and W_n for each $n = 1,2,3...$ exists and are unique.

Using this scheme, we construct two monotone sequences $\{V_n(t)\}$ and $\{W_n(t)\}$. We now claim that these sequences satisfy the relation

$$V_0 \le V_1 \le \ldots \le V_n \le W_n \le \ldots \le W_1 \le W_0 \text{ on } I$$

To prove our claim first we have to show that

$$V_0 \le V_1 \le W_1 \le W_0.$$

For this Consider $P = V_0 - V_1$

$$\begin{aligned} \text{Then} \quad P' &= V_0' - V_1' \\ &\le F_1(t,V_0) + F_2(t,W_0) - F_1(t,V_0) - F_2(t,W_0) = 0 \end{aligned}$$

and $P(0) = V_0(0) - V_1(0) \le 0$. Thus the hypothesis of Corollary-2 is satisfied and we conclude that $P(t) \le 0$ on I, which implies $V_0 \le V_1$ on I. Similarly we prove $W_1 \le W_0$. To show that $V_1 \le W_1$

$$\begin{aligned} \text{Set} \quad R &= V_1 - W_1 \\ \text{Then} \quad R' &= V_1' - W_1' \\ &\le F_1(t,V_0) + F_2(t,W_0) - F_1(t,W_0) - F_2(t,V_0) \le 0 \end{aligned}$$

and $R(0) = V_1(0) - W_1(0) \le 0$.

Hence by Corollary-2 we have $R(t) \le 0$ on I that is, $V_1(t) \le W_1(t)$, on I. Thus the claim $V_0 \le V_1 \le W_1 \le W_0$ on I is proved.

Assume that $V_{k-1} \le V_k \le W_k \le W_{k-1}$ on I for $k > 1$, where V_{k-1}, V_k are the lower solutions of the IVP (13)-(14) and W_{k-1}, W_k are the upper solutions of the IVP (15)-(16) for $n = k-1$, $n = k$ respectively.

Now we claim that the following relation holds.

$$V_k \le V_{k+1} \le W_{k+1} \le W_k \text{ on } I$$

To prove this

$$\begin{aligned} \text{Put} \quad S &= V_k - V_{k+1} \\ \text{Then} \quad S' &= V_k' - V_{k+1}' \\ &\le F_1(t,V_{k-1}) + F_2(t,W_{k-1}) - F_1(t,V_k) - F_2(t,W_k) \le 0 \end{aligned}$$

Further, $S(0) = V_k(0) - V_{k+1}(0) = 0$. Hence by Corollary-2 yields that $S(t) \le 0$ and consequently, $V_k(t) \le V_{k+1}(t)$, on I. In a similar manner we can prove $W_{k+1}(t) \le W_k(t)$ on I.

Next to prove $V_{k+1}(t) \le W_{k+1}(t)$, on I,

$$\begin{aligned} &\text{Set} \quad T = V_{k+1} - W_{k+1} \\ &\text{Then} \quad T' = V'_{k+1} - W'_{k+1} \\ &\qquad \le F_1(t,V_k) + F_2(t,W_k) - F_1(t,W_k) - F_2(t,V_k) \le 0 \end{aligned}$$

$T(0) = V_{k+1}(0) - W_{k+1}(0) = 0$. Hence by Corollary-2 yields that $T(t) \le 0$.

$$\text{Hence } V_k \le V_{k+1} \le W_{k+1} \le W_k \text{ on } I.$$

Thus we obtain two monotone sequences $\{V_n\}$ and $\{W_n\}$ satisfying

$$V_0 \le V_1 \le \ldots \le V_n \le W_n \le \ldots \le W_1 \le W_0 \text{ on } I$$

Now we can easily conclude that the sequence $\{V_n\}$ is monotone nondecreasing and bounded above and hence is convergent to some limit function ρ. Also $\{W_n\}$ is nonincreasing and bounded below and is convergent to some limit function r. Thus the minimal solution ρ and maximal solution ρ of IVP (10)-(11) exist and satisfy the relation

$$V_0 \le V_1 \le \ldots \le V_n \le \rho \le r \le W_n \le \ldots \le W_1 \le W_0 \text{ on } I.$$

Further to prove that these minimal solution ρ and maximal solution r are coupled qusisolutions of nonlinear BVP (1)-(2). It is enough to show that ρ and r are satisfy the following nonlinear boundary conditions

$$g(r(0),\rho(T)) = 0 = g(\rho(0),r(T)).$$

We know that the function $g(x,.)$ is nonincreasing in x. So we have

$$g(\rho(0),r(T)) \le g(V_0(0),r(T)), \text{ for } V_0(0) \le \rho(0) \tag{17}$$

and also

$$g(\rho(0),r(T)) \ge g(W_0(0),r(T)), \text{ for } \rho(0) \le W_0(0). \tag{18}$$

Further $g(.,y)$ is nondecreasing in y

$$g\left(V_0(0), r(T)\right) \leq g\left(V_0(0), W_o(T)\right), \text{ for } r(T) \leq W_0(T) \tag{19}$$

and also

$$g\left(W_0(0), r(T)\right) \geq g\left(W_0(0), V_0(T)\right), \text{ for } V_0(T) \leq r(T) \tag{20}$$

and hence from (17) and (19) we get

$$g\left(\rho(0), r(T)\right) \leq g\left(V_0(0), r(T)\right) \leq g\left(V_0(0), W_0(T)\right) \leq 0 \tag{21}$$

Now by using (18) and (20) we obtain

$$g\left(\rho(0), r(T)\right) \geq g\left(W_0(0), r(T)\right) \geq g\left(W_0(0), V_0(T)\right) \geq 0 \tag{22}$$

Now from the inequalities (21) and (22) we get $g\left(\rho(0), r(T)\right) = 0.$ Similarly we can prove $g\left(r(0), \rho(T)\right) = 0.$ We conclude that condition (8) holds. Thus ρ and r are coupled quasisolutions of the nonlinear boundary value problem (1)-(2) with $\rho \leq r.$ To complete the proof, it is enough to show that $\rho \leq r$.

$$\begin{aligned}
\text{Consider } \phi(t) &= r(t) - \rho(t) \\
\text{Then } \phi' &= r'(t) - \rho'(t) \\
&= F_1(t,r) - F_2(t,\rho) - F_1(t,\rho) - F_2(t,r) \\
&= F_1(t,r) - F_1(t,\rho) - [F_2(t,r) - F_2(t,\rho)] \\
&= L_1(r-\rho) + L_2(r-\rho) \\
&= (L_1 + L_2)(r-\rho) \\
&= M(r-\rho) \leq M\phi.
\end{aligned}$$

and $\phi(0) = 0 = r(0) - \rho(0) = 0.$ Applying Corollary-1 we have $r(t) \leq \rho(t)$ on I . Therefore the coupled quasisolutions satisfy the relation $\rho = r.$ Hence we conclude that $\rho(t) = r(t)$ is the unique solution of nonlinear BVP (1)-(2).

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