Novel solution of nonlinear time fractional Rosenau-Hyman equation

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K. Raghavendar conceptualized the idea and developed the code. All authorscollaborated in writing and reviewing the manuscript.

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The Rosenau-Hyman equation models the nonlinear dispersion in pattern formation in liquid droplets. In this article, the nonlinear time-fractional Rosenau-Hyman equation is analysed using a dependable technique that combines natural transformation and decomposition methods. The fractional derivative is taken into consideration with a well-known singular kernel derivative known as the Caputo derivative, in addition to two nonsingular kernel derivatives known as the Atangana-Baleanu-Caputo derivative and the Caputo-Fabrizio derivative. The dominant framework accurately represents the way in which the outcomes behave when represented by different fractional orders. The convergence analysis was conducted to illustrate the uniqueness of the solutions. In order to verify the efficacy of the techniques being evaluated, numerical simulations are provided. It can be concluded, on the basis of the study's results, that the methodology for investigating fractional differential equations is both effective and reliable. The obtained results are compared numerically and graphically to the exact solutions. The proposed approach would be highly effective for solving a variety of fractional partial differential equations.

Keywords: Atangana-Baleanu-Caputo derivative; Caputo derivative; Caputo-Fabrizio derivative; Natural transform decomposition method; Time fractional Rosenau-Hyman equation

1.1 Introduction

The theory of fractional derivatives and integral operators has driven an excellent interest among scientists for its various applications and significance in economics, biology, mathematics, finance, and physics. Through the use of fractional calculus, mathematical models, coupled linear and nonlinear equations with initial and boundary conditions, used in a variety of fields and technologies, can be expanded upon and described more broadly [1, 2, 3, 4, 5, 6, 7, 8]. In the last few decades, several fractional derivative formulations have appeared. Liouville and Riemann created the formulation for the fractional derivative at the end of the nineteenth century. However, Leibniz and L'Hospital had already addressed the non-integer derivative and integral in 1695. The

Riemann-Liouville (R-L) derivative concept was developed by Riemann in the year 1876. Since then, several branches of scientific and technological sectors have shown how useful these fractional derivatives and integrals of this R-L type are. A Michele Caputo (C) derivative is a special type of fractional derivative that he suggested for a R-L fractional integral around the turn of the 20th century. Atangana and Baleanu effectively applied the newly developed non-singular fractional operator from Caputo and Fabrizio to solve a range of difficult phenomena, including biological ones. Fractional calculus was established on formal foundations by several notable mathematicians, including Euler, Liouville, Abel, Grunwald, Heaviside, Riemann, Lagrange, and Fourier. Recently, many research scholars have focused on various analytical and semi-analytical techniques. For instance, these methods include the homotopy perturbation transform method [9], the reduced differential transform method [10], the Laplace transform method and variational iteration method [11], the q-homotopy analysis transform method [12], the Laplace Adomian decomposition method [13], the reduced power series method [14], and the homotopy analysis transform method [15].

Gilson and Pickering introduced the Gilson-Pickering equation in 1995. The nonlinear third order partial differential equation (PDE) of the type can be modelled by the Gilson-Pickering equation of the form [16].

$$u_{\tau} - \rho u_{\zeta\zeta\tau} + 2\gamma u_{\zeta} - u u_{\zeta\zeta\zeta} - \beta u u_{\zeta} - \theta u_{\zeta} u_{\zeta\zeta} = 0, \quad (1)$$

where $\rho, \gamma, \beta, \theta$ are the constants. Three possible exceptional cases exist for (1) in the above equation. In equation (1), when $\rho = 1, \gamma = 0.5, \beta = -1, \theta = 3$ becomes the Fornberg–Whitham equation. In equation (1), when $\rho = 1, \gamma = 0, \beta = -3, \theta = 2$ becomes the Fuchs Steiner–Fokas–Camassa– Holm equation. In the above equation (1), when

$\rho = 0, \gamma = 0, \beta = 1, \theta = 3$ becomes the Rosenau-Hyman equation (RHE). Consider the time-fractional Rosenau-Hyman equation (TFRHE) of the form:

$$D_{\tau}^{\mu} u = u u_{\zeta\zeta\zeta} + u u_{\zeta} + 3 u_{\zeta} u_{\zeta\zeta}, \quad (2)$$

with the initial condition

$$u(\zeta, 0) = -\frac{8}{3} c \cos^2\left(\frac{\zeta}{4}\right). \quad (3)$$

Where c is a constant and μ is the order of the fractional derivatives $(0 < \mu \leq 1)$. This equation bears the names of Philip Rosenau and James M. Hyman, who wrote the 1993 Compactions inquiry [17]. The RHE is a two-parameter nonlinear PDE modelled to study the nonlinear behavior of dispersion in an organized pattern formation in liquid drops. It has numerous applications in many fields of science and technology. Due to its tremendous applications in various fields, it has attracted the attention of numerous researchers. Recently, a number of studies have investigated the TFRHE using numerical and approximation techniques, including the q-homotopy analysis method [18], the reduced differential transform method (RDTM) and q-homotopy analysis transform method [19], and the variational iteration method (VIM) and homotopy perturbation method (HPM) [20].

To the best of our knowledge, no attempt has been made to solve the time-fractional Rosenau-Hyman equation using the natural transform decomposition method (NTDM). The main aim of this article is to study the approximate solutions of nonlinear TFRHE using the NTDM by considering the fractional derivative in the C, CF, and ABC. The proposed method uses a unique integral transform, namely the natural transform (NT), in combination with the Adomain decomposition approach. The novel method always yields an exact or approximate solution in the form of a fast convergence series. The proposed methodology effectively addresses fractional nonlinear equations, obviating the requirement of a Lagrange multiplier as seen in the variational iteration method or Adomian polynomials as employed in the Adomian decomposition method. These techniques avoid the occurrence of round-off errors by eliminating the need for linearization, specified assumptions, perturbation, or discretization. Recently, numerous scholars have investigated the various physical problems using NTDM, such as the Kawahara and modified Kawahara equations [21], Klein-Gordon equation [22], Burgers-Huxley equation [23], Swift-Hohenberg equation [24], and coupled fractional Ramani equations [25].

The article is organized in the following manner: The fundamental definitions used in this study are presented in Section 1.2. Section 1.3 provides the NTDM solution process for the equation under consideration. In Section 1.4, the uniqueness and convergence results of the suggested technique are established. The approximate solutions of the TFRHE are given in Section 1.5. In Section 1.6, the results and discussion of the current study are presented. The conclusions of this study are presented in Section 1.7.

1.2 Basic Definitions

In this section, we have presented definitions for C, CF, and ABC derivatives and the natural transform of these derivatives.

1.2.1 Definition [26]

The C derivative the function $h(\tau) \in C_{-1}^{q}, q \in \mathbb{N}$ of order μ is as follows

$$D^{\mu} h(\tau) = \frac{1}{\Gamma(q-\mu)} \int_{0}^{\tau} (\tau-\xi)^{q-\mu-1} h^{q}(\xi) d\xi, \tau > 0, q-1 < \mu \leq q. \tag{4}$$

1.2.2 Definition [27]

The fractional derivative CF of the function $h(\tau)$ is defined as

$$ {}^{CF}D_{\tau}^{\delta}h(\tau)=\frac{1}{1-\mu}\int_{0}^{\tau}h^{1}(\xi)\exp\left(\frac{-\mu(\tau-\xi)}{1-\mu}\right)d\xi. \tag{5} $$

1.2.3 Definition [28]

The definition of fractional ABC derivative of the function $h(\tau)$ is defined as

$$ {}^{ABC}D_{\tau}^{\mu}h(\tau)=\frac{M[\mu]}{1-\mu}\int_{0}^{\tau}h^{1}(\xi)E_{\mu}\left(\frac{-\mu(\tau-\xi)^{\mu}}{1-\mu}\right)d\xi, 0<\mu<1. \tag{6} $$

Where E_{μ} is the Mittag-Leffler function.

1.2.4 Definition [29]

NT of the function $h(\tau)$ is defined as

$$ N^{+}[h(\tau)]=R(s,u)=\frac{1}{u}\int_{0}^{\infty}e^{(\frac{-s\tau}{u})}h(\tau)d\tau, u,s\in(0,\infty). \tag{7} $$

1.2.5 Definition [30]

NT of C derivative is defined as

$$ N^{+}[{}_{0}^{C}D_{\tau}^{\mu}u(\tau)]=\left(\frac{s}{v}\right)^{\mu}\left(N^{+}[u(\tau)]-\frac{1}{s}u(0)\right). \tag{8} $$

1.2.6 Definition [31]

The definition of the NT for CF derivative is

$$ N^{+}[{}_{0}^{CF}D_{\tau}^{\mu}u(\tau)]=\frac{1}{f(\mu,s,v)}\left(N^{+}[u(\tau)]-\frac{1}{s}u(0)\right), \tag{9} $$

where $f(\mu,s,v)=1-\mu+\mu\left(\frac{v}{s}\right)$.

1.2.7 Definition [30]

NT of ABC derivative is defined as

$$N^{+}[{}^{ABC}_{\ 0}D^{\mu}_{\tau}u(\tau)]=\frac{1}{g(\mu,s,v)}\left(N^{+}[u(\tau)]-\frac{1}{s}u(0)\right), \quad (10)$$

where $g(\mu,s,v)=\dfrac{1-\mu+\mu\left(\frac{v}{s}\right)^{\mu}}{M[\mu]}$, and $M[\mu]$ is a normalization function.

1.3 Basic Idea of NTDM

In this section, we present the proposed method for (2) along with the initial condition (3) utilizing singular and nonsingular kernel derivatives.

NTDM$_C$**:**In view of equation (8) and initial condition (3), we obtain

$$\left(\frac{s}{v}\right)^{\mu}\left[N^{+}(u(\zeta,\tau))+\frac{\frac{8}{3}c\cos^{2}\left(\frac{\zeta}{4}\right)}{s}\right]=N^{+}\left[uu_{\zeta\zeta\zeta}+uu_{\zeta}+3u_{\zeta}u_{\zeta\zeta}\right]. \quad (11)$$

Taking the inverse NT on equation (11), we have

$$u(\zeta,\tau)=N^{-1}\left[-\frac{\frac{8}{3}c\cos^{2}\left(\frac{\zeta}{4}\right)}{s}+\left(\frac{v}{s}\right)^{\mu}N^{+}\left[uu_{\zeta\zeta\zeta}+uu_{\zeta}+3u_{\zeta}u_{\zeta\zeta}\right]\right]. \quad (12)$$

The decomposition of the nonlinear terms is as follows:

$$uu_{\zeta\zeta\zeta}=\sum_{k=0}^{\infty}A_{k},$$
$$uu_{\zeta}=\sum_{k=0}^{\infty}B_{k},$$
$$u_{\zeta}u_{\zeta\zeta}=\sum_{k=0}^{\infty}C_{k}, \quad (13)$$

where A_k, B_k and C_k are the Adomian polynomials [32].

$u(\zeta,\tau)$ have the infinite series solution of the form

$$u(\zeta,\tau)=\sum_{k=0}^{\infty}u_{k}(\zeta,\tau). \quad (14)$$

Making replacements of equations (13) and (14) into equation (12), we have

$$\sum_{k=0}^{\infty} u_k(\zeta,\tau) = -N^{-1}\left[\frac{\frac{8}{3}c\cos^2\left(\frac{\zeta}{4}\right)}{s}\right] + N^{-1}\left[\begin{array}{l}\left(\frac{v}{s}\right)^{\mu} \\ N^{+}\left[\begin{array}{l}\sum_{k=0}^{\infty}\sum_{j=0}^{k} u_j (u_{k-j})_{\zeta\zeta\zeta} \\ +\sum_{k=0}^{\infty}\sum_{j=0}^{k} u_j (u_{k-j})_{\zeta} + 3\sum_{k=0}^{\infty}\sum_{j=0}^{k} (u_{k-j})_{\zeta} (u_{k-j})_{\zeta\zeta}\end{array}\right]\end{array}\right].$$

(15)

From equation (15), we have

$$\begin{aligned}
{}^{C}u_0(\zeta,\tau) &= -N^{-1}\left[\frac{\frac{8}{3}c\cos^2\left(\frac{\zeta}{4}\right)}{s}\right], \\
{}^{C}u_1(\zeta,\tau) &= N^{-1}\left[\left(\frac{v}{s}\right)^{\mu} N\left[u_0 u_{0\zeta\zeta\zeta} + u_0 u_{0\zeta} + 3u_0 u_{0\zeta\zeta}\right]\right], \\
{}^{C}u_2(\zeta,\tau) &= N^{-1}\left[\left(\frac{v}{s}\right)^{\mu} N^{+}\left[u_0 u_{1\zeta\zeta\zeta} + u_1 u_{0\zeta\zeta\zeta} + u_0 u_{1\zeta} + u_1 u_{0\zeta} + 3(u_{0\zeta} u_{1\zeta\zeta} + u_{1\zeta} u_{0\zeta\zeta})\right]\right], \\
&\vdots \\
{}^{C}u_{k+1}(\zeta,\tau) &= N^{-1}\left[\left(\frac{v}{s}\right)^{\mu} N^{+}\left[\sum_{j=0}^{k} u_j (u_{k-j})_{\zeta\zeta\zeta} + \sum_{j=0}^{k} u_j (u_{k-j})_{\zeta} + 3\sum_{j=0}^{k} (u_{k-j})_{\zeta} (u_{k-j})_{\zeta\zeta}\right]\right], k \geq 0.
\end{aligned}$$

(16)

Making replacements of equation (16) into equation (14), we obtain the series solution as

$${}^{C}u(\zeta,\tau) = {}^{C}u_0(\zeta,\tau) + {}^{C}u_1(\zeta,\tau) + {}^{C}u_2(\zeta,\tau) + \cdots . (17)$$

NTDM$_{CF}$**:** In view of equation (9) and the initial condition (3), we obtain

$$\frac{1}{f(\mu,s,v)}\left[N^{+}(u(\zeta,\tau)) + \frac{\frac{8}{3}c\cos^2\left(\frac{\zeta}{4}\right)}{s}\right] = N^{+}\left[uu_{\zeta\zeta\zeta} + uu_{\zeta} + 3u_{\zeta}u_{\zeta\zeta}\right].$$

(18)

Taking inverse NT on equation (18), we have

$$u(\zeta,\tau)=N^{-1}\left[-\frac{\frac{8}{3}c\cos^2\left(\frac{\zeta}{4}\right)}{s}+f(\mu,s,v)N^{+}\left[uu_{\zeta\zeta\zeta}+uu_{\zeta}+3u_{\zeta}u_{\zeta\zeta}\right]\right]. \quad (19)$$

Now, by substituting equations (13) and (14) into equation (19) to obtain

$$\sum_{k=0}^{\infty}u_k(\zeta,\tau)=-N^{-1}\left[\frac{\frac{8}{3}c\cos^2\left(\frac{\zeta}{4}\right)}{s}\right]+N^{-1}\left[\begin{array}{l} f(\mu,s,v) \\ N^{+}\left[\begin{array}{l}\sum_{k=0}^{\infty}\sum_{j=0}^{k}u_j(u_{k-j})_{\zeta\zeta\zeta} \\ +\sum_{k=0}^{\infty}\sum_{j=0}^{k}u_j(u_{k-j})_{\zeta}+3\sum_{k=0}^{\infty}\sum_{j=0}^{k}(u_{k-j})_{\zeta}(u_{k-j})_{\zeta\zeta}\end{array}\right]\end{array}\right].$$

(20)

From equation (20), we have

$$^{CF}u_0(\zeta,\tau)=-N^{-1}\left[\frac{\frac{8}{3}c\cos^2\left(\frac{\zeta}{4}\right)}{s}\right],$$

$$^{CF}u_1(\zeta,\tau)=N^{-1}\left[f(\mu,s,v)N\left[u_0u_{0\zeta\zeta\zeta}+u_0u_{0\zeta}+3u_0u_{0\zeta\zeta}\right]\right],$$

$$^{CF}u_2(\zeta,\tau)=N^{-1}\left[f(\mu,s,v)N^{+}\left[u_0u_{1\zeta\zeta\zeta}+u_1u_{0\zeta\zeta\zeta}+u_0u_{1\zeta}+u_1u_{0\zeta}+3(u_{0\zeta}u_{1\zeta\zeta}+u_{1\zeta}u_{0\zeta\zeta})\right]\right],$$

$$\vdots$$

$$^{CF}u_{k+1}(\zeta,\tau)=N^{-1}\left[f(\mu,s,v)N^{+}\left[\sum_{j=0}^{k}u_j(u_{k-j})_{\zeta\zeta\zeta}+\sum_{j=0}^{k}u_j(u_{k-j})_{\zeta}+3\sum_{j=0}^{k}(u_{k-j})_{\zeta}(u_{k-j})_{\zeta\zeta}\right]\right],k\geq 0.$$

(21)

Making replacements of equation (21) into equation (14), we obtain the series solution as

$$^{CF}u(\zeta,\tau)={}^{CF}u_0(\zeta,\tau)+{}^{CF}u_1(\zeta,\tau)+{}^{CF}u_2(\zeta,\tau)+\cdots. \quad (22)$$

NTDM$_{ABC}$: In view of equation (10) and the initial conditions (3), we obtain

$$\frac{1}{g(\mu,s,v)}\left[N^{+}(u(\zeta,\tau))+\frac{\frac{8}{3}c\cos^2\left(\frac{\zeta}{4}\right)}{s}\right]=N^{+}\left[uu_{\zeta\zeta\zeta}+uu_{\zeta}+3u_{\zeta}u_{\zeta\zeta}\right]. \quad (23)$$

Taking the inverse NT on equation (23), we have

$$u(\zeta,\tau)=N^{-1}\left[-\frac{\frac{8}{3}c\cos^2\left(\frac{\zeta}{4}\right)}{s}+g(\mu,s,v)N^{+}\left[uu_{\zeta\zeta\zeta}+uu_{\zeta}+3u_{\zeta}u_{\zeta\zeta}\right]\right]. \quad (24)$$

Now, by substituting equations (13) and (14) into equation (24) to obtain

$$\sum_{k=0}^{\infty}u_k(\zeta,\tau)=-N^{-1}\left[\frac{\frac{8}{3}c\cos^2\left(\frac{\zeta}{4}\right)}{s}\right]+N^{-1}\left[\begin{array}{l}g(\mu,s,v)\\ N^{+}\left[\begin{array}{l}\sum_{k=0}^{\infty}\sum_{j=0}^{k}u_j(u_{k-j})_{\zeta\zeta\zeta}\\ +\sum_{k=0}^{\infty}\sum_{j=0}^{k}u_j(u_{k-j})_{\zeta}+3\sum_{k=0}^{\infty}\sum_{j=0}^{k}(u_{k-j})_{\zeta}(u_{k-j})_{\zeta\zeta}\end{array}\right]\end{array}\right].$$

(25)

From equation (25), we have

$${}^{ABC}u_0(\zeta,\tau)=-N^{-1}\left[\frac{\frac{8}{3}c\cos^2\left(\frac{\zeta}{4}\right)}{s}\right],$$

$${}^{ABC}u_1(\zeta,\tau)=N^{-1}\left[g(\mu,s,v)N\left[u_0u_{0\zeta\zeta\zeta}+u_0u_{0\zeta}+3u_0u_{0\zeta\zeta}\right]\right],$$

$${}^{ABC}u_2(\zeta,\tau)=N^{-1}\left[g(\mu,s,v)N^{+}\left[u_0u_{1\zeta\zeta\zeta}+u_1u_{0\zeta\zeta\zeta}+u_0u_{1\zeta}+u_1u_{0\zeta}+3(u_{0\zeta}u_{1\zeta\zeta}+u_{1\zeta}u_{0\zeta\zeta})\right]\right],$$

$$\vdots$$

$${}^{ABC}u_{k+1}(\zeta,\tau)=N^{-1}\left[g(\mu,s,v)N^{+}\left[\sum_{j=0}^{k}u_j(u_{k-j})_{\zeta\zeta\zeta}+\sum_{j=0}^{k}u_j(u_{k-j})_{\zeta}+3\sum_{j=0}^{k}(u_{k-j})_{\zeta}(u_{k-j})_{\zeta\zeta}\right]\right],k\geq 0.$$

(26)

Making replacements of equation (26) into equation (14), we obtain the series solution as

$${}^{ABC}u(\zeta,\tau)={}^{ABC}u_0(\zeta,\tau)+{}^{ABC}u_1(\zeta,\tau)+{}^{ABC}u_2(\zeta,\tau)+\cdots. \quad (27)$$

1.4 Convergence Analysis

In this section, we studied the uniqueness and convergence [30].

1.4.1 Theorem The $NTDM_C$ solution is unique when $0<(\theta_1+\theta_2)\frac{\tau^{\mu}}{\Gamma(\mu+1)}<1.$

1.4.2 Theorem The $NTDM_{CF}$ solution is unique when $0<(\theta_1+\theta_2)(1-\mu+\mu\tau)<1.$

1.4.3 Theorem The $NTDM_{ABC}$ solution is unique when $0<(\theta_1+\theta_2)\left(1-\mu+\mu\frac{\tau^\mu}{\Gamma(\mu+1)}\right)<1.$

1.4.4 Theorem The $NTDM_C$ solution is convergent.

1.4.5 Theorem $NTDM_{CF}$ solution is convergent.

1.4.6 Theorem $NTDM_{ABC}$ solution is convergent.

1.5 Numerical Example of TFRHE

In this section, we have presented the approximate solutions of TFRHE using the NTDM with the help of singular and non-singular kernel derivatives, namely, C, CF, and ABC.

$NTDM_C$: We get the solutions of $NTDM_C$ as,

$$
\begin{aligned}
{}^{C}u_0(\zeta,\tau) &= -\frac{8c}{3}\cos^2\left(\frac{\zeta}{4}\right),\\
{}^{C}u_1(\zeta,\tau) &= -\frac{2c^2}{3}\sin\left(\frac{\zeta}{2}\right)\frac{\tau^\mu}{\Gamma(\mu+1)},\\
{}^{C}u_2(\zeta,\tau) &= \frac{c^3}{3}\cos\left(\frac{\zeta}{2}\right)\frac{\tau^{2\mu}}{\Gamma(2\mu+1)},\\
{}^{C}u_3(\zeta,\tau) &= \frac{c^4}{6}\sin\left(\frac{\zeta}{2}\right)\frac{\tau^{3\mu}}{\Gamma(3\mu+1)},\\
&\vdots
\end{aligned}
$$

we can follow the above NTDMC procedure and get the approximate solution as

$$
{}^{C}u(\zeta,\tau) = -\frac{8c}{3}\cos^2\left(\frac{\zeta}{4}\right)-\frac{2c^2}{3}\sin\left(\frac{\zeta}{2}\right)\frac{\tau^\mu}{\Gamma(\mu+1)}+\frac{c^3}{3}\cos\left(\frac{\zeta}{2}\right)\frac{\tau^{2\mu}}{\Gamma(2\mu+1)}+\frac{c^4}{6}\sin\left(\frac{\zeta}{2}\right)\frac{\tau^{3\mu}}{\Gamma(3\mu+1)}-\cdots.
$$

$NTDM_{CF}$: We get the solutions of $NTDM_{CF}$ as,

$$
\begin{aligned}
&{}^{CF}u_0(\zeta,\tau) = -\frac{8c}{3}\cos^2\left(\frac{\zeta}{4}\right),\\
&{}^{CF}u_1(\zeta,\tau) = -\frac{2c^2}{3}\sin\left(\frac{\zeta}{2}\right)[1+\mu\tau-\mu],\\
&{}^{CF}u_2(\zeta,\tau) = \frac{c^3}{3}\cos\left(\frac{\zeta}{2}\right)\left[2\mu\tau(1-\mu)+\mu^2\frac{\tau^2}{\Gamma(2)}+(1-\mu)^2\right],\\
&{}^{CF}u_3(\zeta,\tau) = \frac{c^4}{6}\sin\left(\frac{\zeta}{2}\right)\left[(1-\mu)^3+3\mu(1-\mu)^2\tau+3\mu^2(1-\mu)\frac{\tau^2}{\Gamma(3)}+\mu^3\frac{\tau^3}{\Gamma(4)}\right],\\
&\vdots
\end{aligned}
$$

we can follow the above $NTDM_{CF}$ procedure and get the approximate solution as

$$
\begin{aligned}
{}^{CF}u(\zeta,\tau) = &-\frac{8c}{3}\cos^2\left(\frac{\zeta}{4}\right)-\frac{2c^2}{3}\sin\left(\frac{\zeta}{2}\right)[1+\mu\tau-\mu]+\frac{c^3}{3}\cos\left(\frac{\zeta}{2}\right)\left[2\mu\tau(1-\mu)+\mu^2\frac{\tau^2}{\Gamma(2)}+(1-\mu)^2\right]\\
&+\frac{c^4}{6}\sin\left(\frac{\zeta}{2}\right)\left[(1-\mu)^3+3\mu(1-\mu)^2\tau+3\mu^2(1-\mu)\frac{\tau^2}{\Gamma(3)}+\mu^3\frac{\tau^3}{\Gamma(4)}\right]+\cdots.
\end{aligned}
$$

$NTDM_{ABC}$: We get the solutions of $NTDM_{ABC}$ as,

$$
\begin{aligned}
&{}^{ABC}u_0(\zeta,\tau) = -\frac{8c}{3}\cos^2\left(\frac{\zeta}{4}\right),\\
&{}^{ABC}u_1(\zeta,\tau) = -\frac{2c^2}{3}\sin\left(\frac{\zeta}{2}\right)\left[\frac{1-\mu}{M[\mu]}+\frac{\mu}{M[\mu]}\frac{\tau^\mu}{\Gamma(\mu+1)}\right],\\
&{}^{ABC}u_2(\zeta,\tau) = \frac{c^3}{3}\cos\left(\frac{\zeta}{2}\right)\left[\left(\frac{1-\mu}{M[\mu]}\right)^2+\frac{2\mu(1-\mu)}{M[\mu]^2}\frac{\tau^\mu}{\Gamma(\mu+1)}+\left(\frac{\mu}{M[\mu]}\right)^2\frac{\tau^{2\mu}}{\Gamma(2\mu+1)}\right],\\
&{}^{ABC}u_3(\zeta,\tau) = \frac{c^4}{6}\sin\left(\frac{\zeta}{2}\right)\left[\left(\frac{1-\mu}{M[\mu]}\right)^3+\frac{3\mu(1-\mu)^2}{M[\mu]^3}\frac{\tau^\mu}{\Gamma(\mu+1)}+\frac{3\mu^2(1-\mu)}{M[\mu]^3}\frac{\tau^{2\mu}}{\Gamma(2\mu+1)}+\left(\frac{\mu}{M[\mu]}\right)^3\frac{\tau^{3\mu}}{\Gamma(3\mu+1)}\right],\\
&\vdots
\end{aligned}
$$

we can follow the above $NTDM_{ABC}$ procedure and get the approximate solution as

$$
\begin{aligned}
{}^{ABC}u(\zeta,\tau) &= -\frac{8c}{3}\cos^2\left(\frac{\zeta}{4}\right) - \frac{2c^2}{3}\sin\left(\frac{\zeta}{2}\right)\left[\frac{1-\mu}{M[\mu]} + \frac{\mu}{M[\mu]}\frac{\tau^\mu}{\Gamma(\mu+1)}\right] \\
&+ \frac{c^3}{3}\cos\left(\frac{\zeta}{2}\right)\left[\left(\frac{1-\mu}{M[\mu]}\right)^2 + \frac{2\mu(1-\mu)}{M[\mu]^2}\frac{\tau^\mu}{\Gamma(\mu+1)} + \left(\frac{\mu}{M[\mu]}\right)^2\frac{\tau^{2\mu}}{\Gamma(2\mu+1)}\right] \\
&+ \frac{c^4}{6}\sin\left(\frac{\zeta}{2}\right)\left[\left(\frac{1-\mu}{M[\mu]}\right)^3 + \frac{3\mu(1-\mu)^2}{M[\mu]^3}\frac{\tau^\mu}{\Gamma(\mu+1)} + \frac{3\mu^2(1-\mu)}{M[\mu]^3}\frac{\tau^{2\mu}}{\Gamma(2\mu+1)} + \left(\frac{\mu}{M[\mu]}\right)^3\frac{\tau^{3\mu}}{\Gamma(3\mu+1)}\right] + \cdots.
\end{aligned}
$$

For $\mu=1$, the $NTDM$ solution approaches the exact solution

$u(\zeta,\tau) = -\frac{8c}{3}\cos^2\left(\frac{\zeta - c\tau}{4}\right)$, where $|(\zeta - c\tau)| \le 2\pi.$

Table 1.1 Comparison of the absolute errors of TFRHE with $\mu=1$ and $c=1$.

ζ	τ	$NTDM_C$	$NTDM_{CF}$	$NTDM_{ABC}$	RDTM[33]
$\frac{\pi}{4}$	0.	0.0001	0.0001	0.0001	0.0000
	2	0.0006	0.0006	0.0006	0.0006
	0.	0.0019	0.0019	0.0019	0.0019
	4				
	0.				
	6				
$\frac{\pi}{2}$	0.	0.0002	0.0002	0.0002	0.0001
	2	0.0012	0.0012	0.0012	0.0011
	0.	0.0039	0.0039	0.0039	0.0039
	4				
	0.				
	6				
$\frac{3\pi}{4}$	0.	0.0002	0.0002	0.0002	0.0002

	2	0.0016	0.0016	0.0016	0.0016
	0.	0.0053	0.0053	0.0053	0.0054
	4				
	0.				
	6				
π	0.	0.0002	0.0002	0.0002	0.0003
	2	0.0018	0.0018	0.0018	0.0018
	0.	0.0060	0.0060	0.0060	0.0059
	4				
	0.				
	6				

Table 1.2 The approximate solution of TFRHE when $\mu=1$ and $c=1$.

ζ	τ	$NTDM_C$	$NTDM_{CF}$	$NTDM_{ABC}$	VIM[20]	HPM[20]
$\frac{\pi}{4}$	0.	-2.6100	-2.6100	-2.6100	-2.6099	-2.6099
	2	-2.6609	-2.6609	-2.6609	-2.6609	-2.6609
	0.	-2.6590	-2.6590	-2.6590	-2.6590	-2.6589
	6					
	1.					
	0					
$\frac{\pi}{2}$	0.	-2.3656	-2.3656	-2.3656	-2.3655	-2.3655
	2	-2.5127	-2.5127	-2.5127	-2.5126	-2.5126
	0.	-2.6128	-2.6128	-2.6128	-2.6127	-2.6127

	6					
	1.					
	0					
$\frac{3\pi}{4}$	0.	-1.9640	-1.9640	-1.9640	-0.4893	-0.4893
	2	-2.1848	-2.1848	-2.1848	-0.7112	-0.7112
	0.	-2.3717	-2.3717	-2.3717	-0.9579	-0.9579
	6					
	1.					
	0					
π	0.	-1.4664	-1.4664	-1.4664	-1.4664	-1.4664
	2	-1.7274	-1.7274	-1.7274	-1.7273	-1.7273
	0.	-1.9726	-1.9726	-1.9726	-1.9725	-1.9725
	6					
	1.					
	0					

Table 1.3 Absolute error of the TFRHE when $\mu = 1$ and $c = 0.1$.

ζ	τ	$NTDM_C$	$NTDM_{CF}$	$NTDM_{ABC}$	Absolute error
0.3	0.3	$1.0930e-08$	$1.0930e-08$	$1.0930e-08$	$1.391e-06$
0.3	1.0	$3.8072e-07$	$3.8072e-07$	$3.8072e-07$	$3.610e-06$
0.3	1.6	$1.4748e-06$	$1.4748e-06$	$1.4748e-06$	$5.820e-06$
1.0	0.3	$3.5710e-08$	$3.5710e-08$	$3.5710e-08$	$8.925e-11$
1.0	1.0	$1.3011e-06$	$1.3011e-06$	$1.3011e-06$	$8.925e-11$
1.0	1.6	$5.2534e-06$	$5.2534e-06$	$5.2534e-06$	$8.925e-11$
1.6	0.3	$5.3605e-08$	$5.3605e-08$	$5.3605e-08$	$7.248e-07$
1.6	1.0	$1.9682e-06$	$1.9682e-06$	$1.9682e-06$	$2.075e-05$
1.6	1.6	$8.0008e-06$	$8.0008e-06$	$8.0008e-06$	$4.016e-05$

Table 1.4 Relative error of the TFRHE when $\mu = 1$ and $c = 0.1$.

					Relative error
0.	0.	$4.1173e-08$	$4.1173e-08$	$4.1173e-08$	$5.245e-06$
3	3	$1.4313e-06$	$1.4313e-06$	$1.4313e-06$	$1.358e-05$
0.	1.	$5.5372e-06$	$5.5372e-06$	$5.5372e-06$	$2.186e-05$
3	0	$1.4211e-07$	$1.4211e-07$	$1.4211e-07$	$3.550e-10$
0.	1.	$5.1347e-06$	$5.1347e-06$	$5.1347e-06$	$3.522e-10$
3	6	$2.0595e-05$	$2.0595e-05$	$2.0595e-05$	$3.497e-10$
1.	0.	$2.3547e-07$	$2.3547e-07$	$2.3547e-07$	$3.227e-06$
0	3	$8.5244e-06$	$8.5244e-06$	$8.5244e-06$	$9.110e-05$
1.	1.	$3.4254e-$	$3.4254e-$	$3.4254e-$	$1.740e-04$
0	0	05	05	05	
1.	1.				
0	6				
1.	0.				
6	3				
1.	1.				
6	0				
1.	1.				
6	6				

Table 1.5 Approximate solution of TFRHE with different values of and c = 1.

	$\mu = 0.25$			$\mu = 0.50$		
	$NTDM_C$	$NTDM_{CF}$	$NTDM_{ABC}$	$NTDM_C$	$NTDM_{CF}$	$NTDM_{ABC}$
0.	-	-2.4971	-2.4971	-	-2.4529	-2.4529

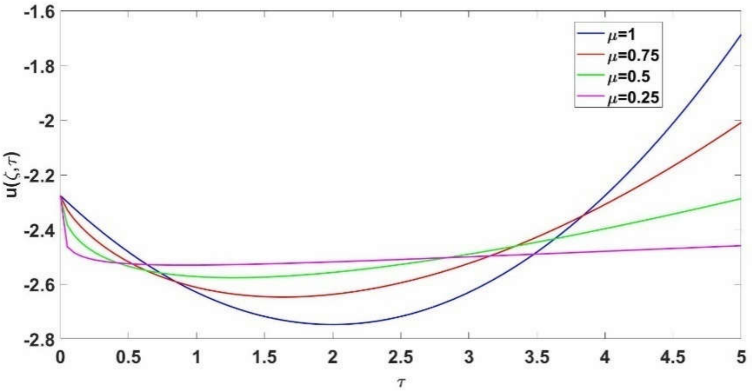
0	2.2761	-2.5100	-2.5127	2.2761	-2.5045	-2.5175
0.	-	-2.5192	-2.5130	-	-2.5413	-2.5270
5	2.5254	-2.5247	-2.5127	2.5344	-2.5634	-2.5274
1.	-			-		
0	2.5303			2.5724		
1.	-			-		
5	2.5260			2.5741		
0.	-	-2.5833	-2.5833	-	-2.6157	-2.6157
0	2.5652	-2.5550	-2.5200	2.5652	-2.5929	-2.5562
0.	-	-2.5220	-2.5045	-	-2.5508	-2.5089
5	2.5561	-2.4841	-2.4936	2.6148	-2.4895	-2.4637
1.	-			-		
0	2.4991			2.5451		
1.	-			-		
5	2.4511			2.4558		
0.	-	-2.2338	-2.2338	-	-2.1196	-2.1196
0	1.8436	-2.2858	-2.3259	1.8436	-2.2378	-2.2985
0.	-	-2.3359	-2.3419	-	-2.3479	-2.3633
5	2.3132	-2.3840	-2.3523	2.2712	-2.4500	-2.4093
1.	-			-		
0	2.3792			2.4110		
1.	-			-		
5	2.4193			2.5034		

π	0.	-	-1.8333	-1.8333	-	-1.6667	-1.6667
	0	1.3333	-1.9167	-1.9880	1.3333	-1.8333	-1.9326
	0.	-	-2.0000	-2.0172	-	-2.0000	-2.0428
	5	1.9518	-2.0833	-2.0368	1.8653	-2.1667	-2.1273
	1.	-			-		
	0	2.0688			2.0856		
	1.	-			-		
	5	2.1473			2.2547		

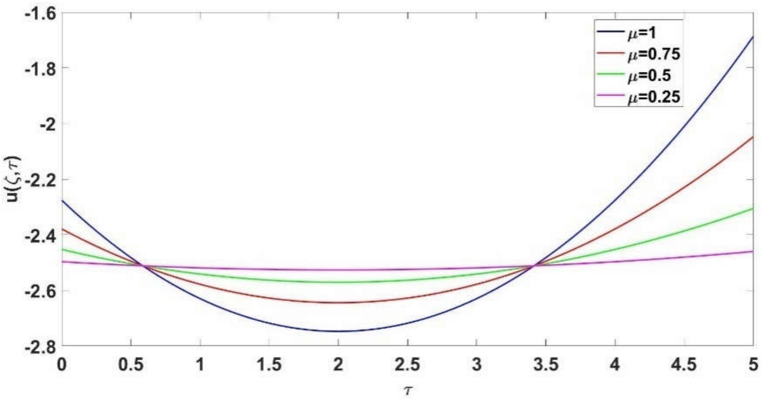
Table 1.6 Approximate solution of TFRHE with different values of and c = 1.

	$\mu = 0.75$			$\mu = 1$		
	$NTDM_C$	$NTDM_{CF}$	$NTDM_{ABC}$	$NTDM_C$	$NTDM_{CF}$	$NTDM_{ABC}$
0.	-2.2761	-2.3793	-	-2.2761	-2.2761	-2.2761
0	-2.5184	-2.4953	2.3793	-2.4824	-2.4824	-2.4824
0.	-2.6118	-2.5781	-	-2.6297	-2.6297	-2.6297
5	-2.6456	-2.6279	2.5156	-2.7181	-2.7181	-2.7181
1.			-			
0			2.5680			
1.			-			
5			2.5871			
0.	-2.5652	-2.6097	-	-2.5652	-2.5652	-2.5652

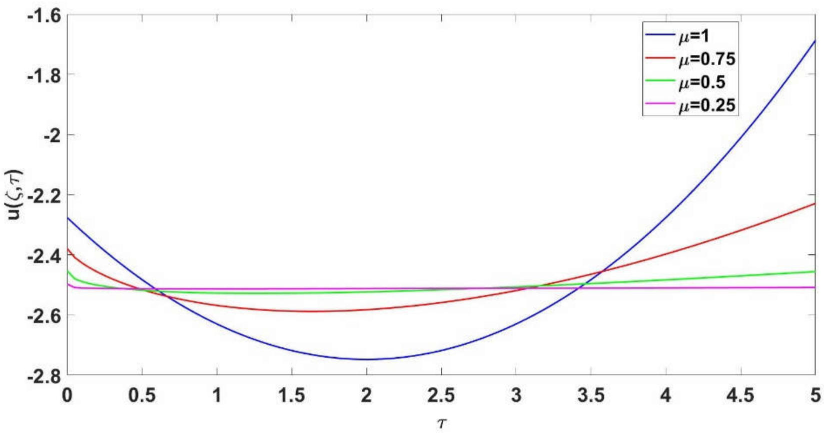
	0	-2.6483	-2.6260	2.6097	-2.6542	-2.6542	-2.6542
	0.	-2.6111	-2.5989	-	-2.6663	-2.6663	-2.6663
	5	-2.5158	-2.5286	2.6127	-2.6014	-2.6014	-2.6014
	1.			-			
	0			2.5619			
	1.			-			
	5			2.4822			
	0.	-1.8436	-1.9896	-	-1.8436	-1.8436	-1.8436
	0	-2.2081	-2.1877	1.9896	-2.1356	-2.1356	-2.1356
	0.	-2.4178	-2.3678	-	-2.3957	-2.3957	-2.3957
	5	-2.5756	-2.5300	2.2384	-2.6240	-2.6240	-2.6240
	1.			-			
	0			2.3862			
	1.			-			
	5			2.5011			
π	0.	-1.3333	-1.5000	-	-1.3333	-1.3333	-1.3333
	0	-1.7646	-1.7500	1.5000	-1.6667	-1.6667	-1.6667
	0.	-2.0587	-2.0000	-	-2.0000	-2.0000	-2.0000
	5	-2.3165	-2.2500	1.8235	-2.3333	-2.3333	-2.3333
	1.			-			
	0			2.0440			
	1.			-			
	5			2.2374			



$(a) NTDM_C$

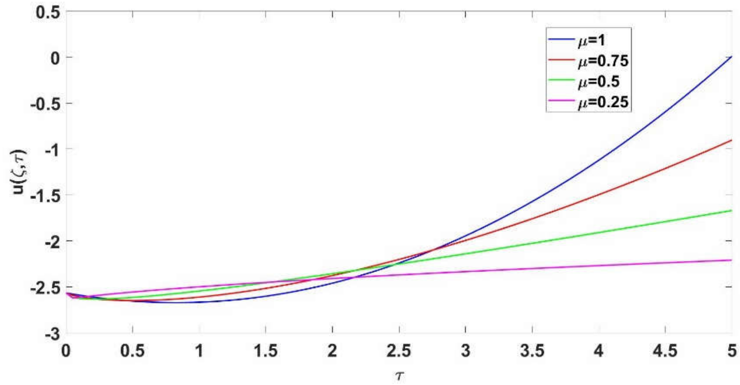


$(b) NTDM_{CF}$

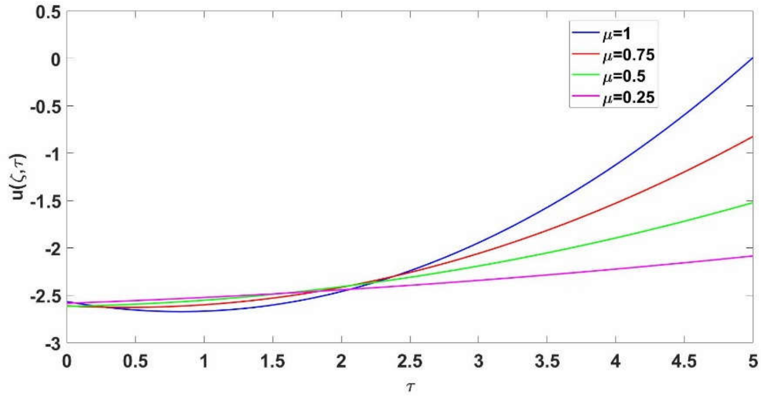


$(c) NTDM_{ABC}$

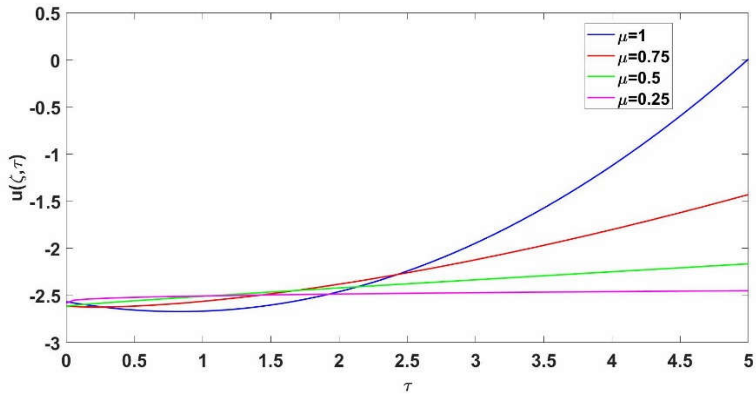
Figure 1.1 Approximate solution of TFRHE at $\zeta = \frac{\pi}{2}$ for different values of μ.



$(a) NDTM_C$

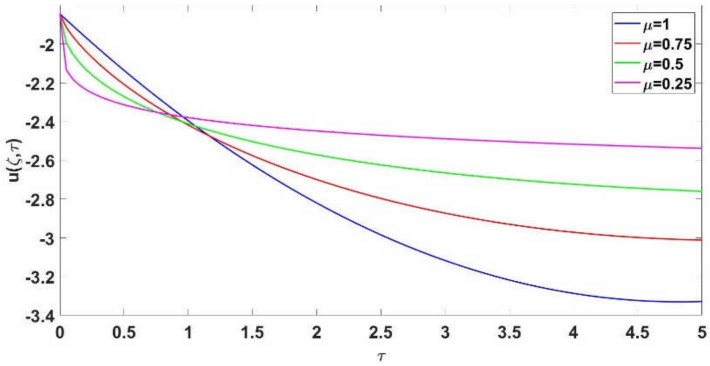


$(b) NTDM_{CF}$

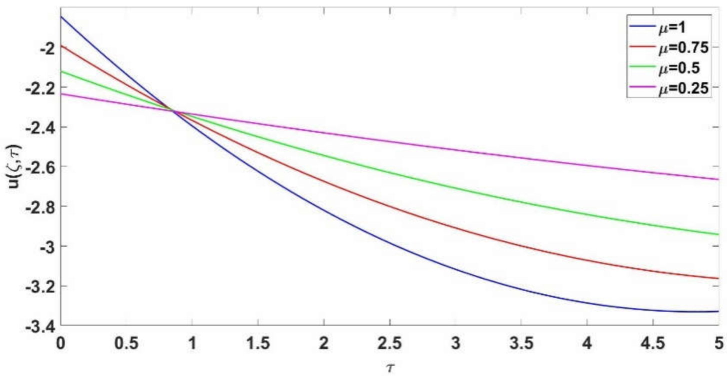


$(c) NTDM_{ABC}$

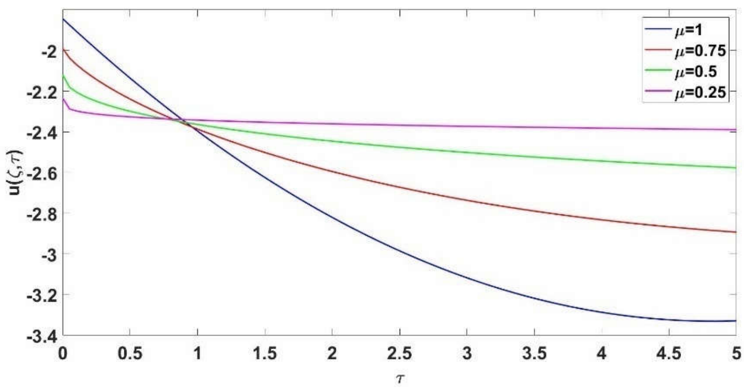
Figure 1.2 Approximate solution of TFRHE at $\zeta = \frac{\pi}{4}$ for different values of μ.



$(a)NTDM_C$

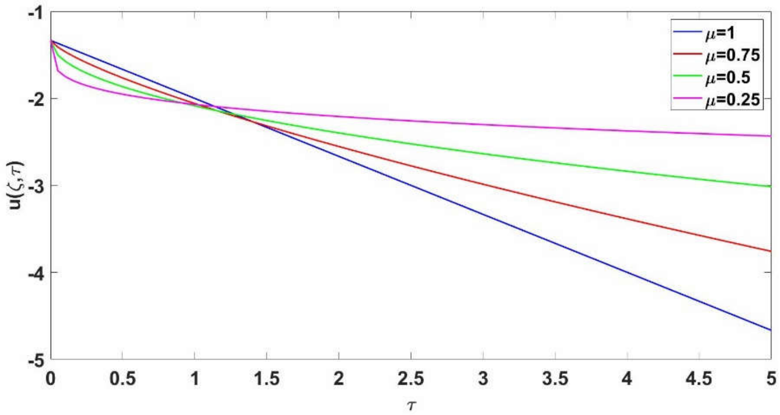


$(b)NTDM_{CF}$

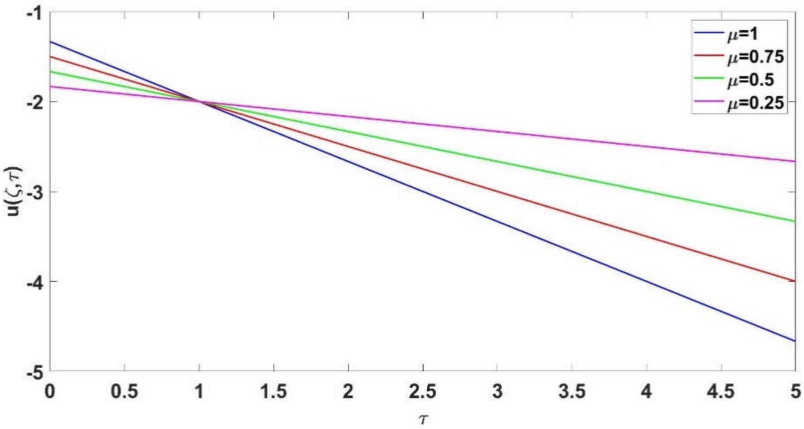


$(c)NTDM_{ABC}$

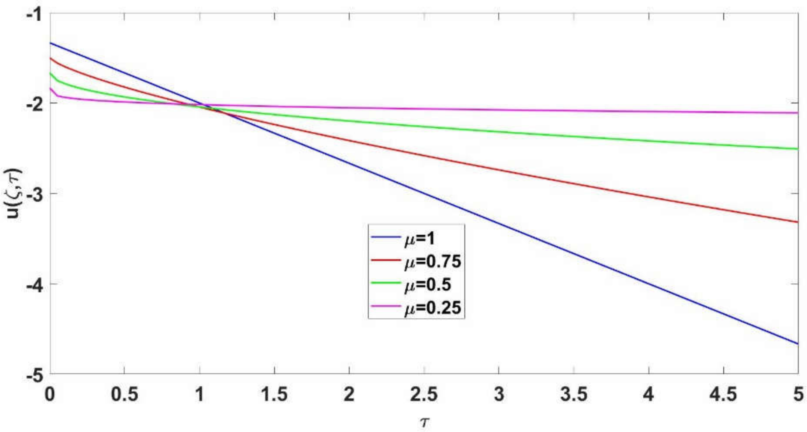
Figure 1.3 Approximate solution of TFRHE at $\zeta = \frac{3\pi}{4}$ for different values of μ.



$(a) NTDM_C$

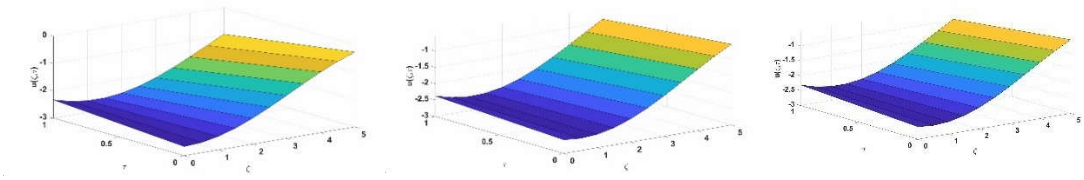


$(b) NTDM_{CF}$

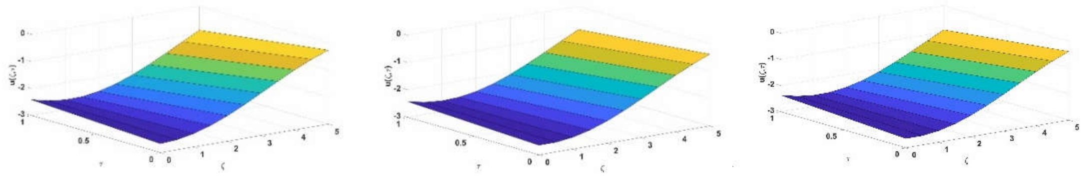


$(c) NTDM_{ABC}$

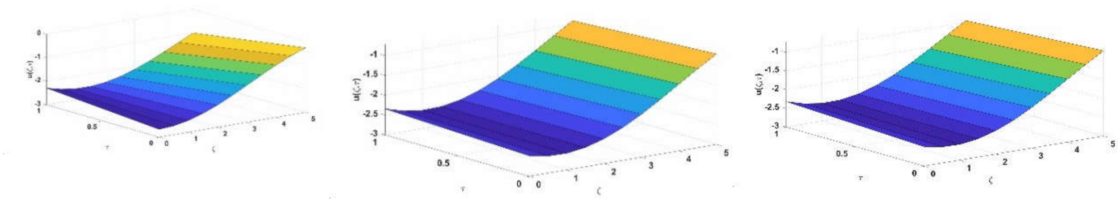
Figure 1.4 Approximate solution of TFRHE at π for different values of μ.



$(a) NTDM_C, \mu = 0.25$ $(b) NTDM_{CF}, \mu = 0.25$ $(c) NTDM_{ABC}, \mu = 0.25$



$(a) NTDM_C, \mu = 0.50$ $(b) NTDM_{CF}, \mu = 0.50$ $(c) NTDM_{ABC}, \mu = 0.50$



$(a) NTDM_C, \mu = 0.75$ $(b) NTDM_{CF}, \mu = 0.75$ $(c) NTDM_{ABC}, \mu = 0.75$

Figure 1.5 Phase plot of ${}^{C}u(\zeta,\tau), {}^{CF}u(\zeta,\tau), {}^{ABC}u(\zeta,\tau)$ at various values of μ **.**

1.6 Results and Discussion

In this results and discussion part, we illustrate the accuracy and efficiency of the proposed technique. Table 1 shows the comparison of the absolute error of the proposed method with RDTM for $\mu = 1$ and $c = 1$ at various values ζ, τ. Table 2 shows the comparison of the approximate solutions of this study with VIM and HPM for $\mu = 1$ and $c = 1$ at various values of ζ, τ. Table 3 and Table 4 show the comparison of the presented method's absolute error and relative error, respectively, for $\mu = 1$ and $c = 0.1$ at various values of ζ, τ. Table 5 and Table 6 shows the NTDM solution for the various values of ζ, τ and μ. From the tables, it can be observed that the approximate solutions of the current study show good agreement with those methods existing in the literature. Figures 1–4 demonstrate the wave behavior of the TFRHE using two-dimensional graphical representations with three various fractional operators for various μ,ζ values. Figure 5 displays the phase plot of the approximate solutions for various fractional order values. From the figures, it can be concluded that all three derivate solutions behave the same. From the tables and figures, it can be observed that when the fraction order μ approaches 1, the approximate solutions converge to the exact solutions. We conclude that our approach has very accurate approximations to the precise solution of the relevant equation based on the outcomes of the TFRHE.

1.7 Conclusions

This article utilizes the natural transform decomposition method to investigate the time-fractional Rosenau-Hyman equation. This equation serves as a mathematical model for the nonlinear dispersion observed in the formation of patterns in liquid droplets. The Caputo, CF, and ABC approaches are utilized to analyze the fractional derivative. The results show that the analytical solutions obtained by employing the NTDM method are pretty close to the exact solutions. The tables and graphs show that when $\mu=1$, the approximate solution of the differential equation approaches the exact solution. This study demonstrates the validity and effectiveness of the implemented method by conducting a comparative analysis of the results obtained and those reported in the existing literature such as RDTM [23], VIM [32] and HPM [32]. Therefore, the suggested methodology for obtaining analytical solutions to nonlinear problems is remarkably reliable and effective. The proposed methodology exhibits straightforwardness in its application to study non-linear time fractional partial differential equations. From the outcomes of this study, we can conclude that this strategy has a faster rate of convergence than the other methods. And also, the results demonstrate the procedure's simplicity and usefulness. The proposed method can be used to solve fractional differential equations in various fields of science and engineering.

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