Two Parameter Fitting Method for Singularly Perturbed Parabolic Convection-Diffusion Problems

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Abstract

In this article, a computational method is devised to solve the multi parameter singularly perturbed one dimensional parabolic equation. The backward Euler method is utilized for temporal discretization. For spatial discretization, the trigonometric spline is used and the first order derivatives in the discrete scheme of trigonometric spline are replaced by the finite differences of higher order. To solve the resulting system, a tridiagonal solver is utilized. The proposed numerical scheme is designed to exhibit parameter-uniform convergence and achieve fourth-order accuracy in space and first-order accuracy in time. The proposed scheme is utilized to solve model examples and compare them with existing methods in the literature, in order to validate the effectiveness of the method.

Keywords: Multi parameter; Singularly perturbed; Trigonometric spline; Uniform mesh; Parameter uniform

1.1 Introduction

Due to the inherent characteristics of certain physical phenomena, such as minuscule viscosity in the Navier-Stokes equations, singularly perturbed partial differential equations (SPPDE) are common. They are also involved in the modelling and analysis of heat and mass transfer processes in situations where the rate of reaction is high, and the thermal conductivity and diffusion coefficients are low. Singularly perturbed models have been developed in biology to represent the dynamics of a variety of biological systems. Many real-life applications make use of the diffusive parameter's tiny size.SPPDEs are encountered in various research domains within applied mathematics [1,2,14]. These include applications in the assessment of water quality in river networks, the modelling of oil extraction from underground reservoirs, the analysis of convective heat transport problems with high Peclet numbers, the study of atmospheric pollution, and the investigation of fluid flow at high Reynolds numbers, among others. More recently, many robust numerical techniques have been created for solving SPPDE

[5,11]. Munyakaji and Patidar [16] worked out similar problem by treating a novel fitted operator finite difference method (FDM). Munyakaji [15] developed a reliable FDM for solving a class of SPPDEs with convection and diffusion terms affected by the two parameters. Aziz and Jain [7] analysed using adaptive splines. In their study, Clevaro et al. [3] successfully addressed the SPPDE problem by employing a numerical method that exhibits uniform convergence with respect to the diffusion parameter. Kadalbajoo and Yadaw [8] studied numerical methods for a class of singularly perturbed boundary value problem with two parameters. A parameter uniform numerical scheme for SPPDE is developed by Gemechis and Mesfin [12] with small delay arising in computational neuroscience. For two-parameter initial BVPs with parabolic convection-reaction-diffusion in one dimension, Das and Mehrmann [4] proposed an adaptive FDM.Gowrisankar and Natesan [6] studied robust numerical scheme to solve SPPDE using classical upwind FDM on layer-adapted nonuniform meshes. In their study, Kadalbajoo and Yadaw[9] conducted an investigation on a Ritz-Galerkin finite element method employed for solving a two-parameter SPBVP. Gemechis and Mesfin [13] considered SPPDE with a small delay on convection and reaction terms. They used the Crank Nicolson method in time derivative discretization and the mid-point upwind FDM on piecewise uniform Shishkin mesh for the space variable derivative discretization.It has been observed that the equations under consideration have not been thoroughly studied numerically. Additionally, the schemes developed in existing literature mostly exhibit linear order of convergence. The desire to enhance accuracy and convergence in solving SPPDEs drives our efforts to develop a uniformly convergent numerical scheme of higher order. Here, we proposed a two-parameter fitting trigonometric spline method to solve a SPPDE on uniform mesh. The SPPDE under consideration includes diffusion-convection terms that are influenced by two

small parameters. For the theoretical analysis, the overall error is decomposed into two components, the first arises from the discretization in time and the second from the discretization in space obtained after the temporal discretization. The rest of the paper is laid out as follows. Section 2 lays out the problem in detail. The discretization of the time and space variables is of interest, and a scheme is presented for doing so in sections 3 and 4. Convergence of the procedure is demonstrated in Section 5. Section 6 presents our findings regarding the comparison of several existing methods. In the final section, we draw some conclusions and discuss the implications.

1.2 Problem description

In this analysis, we will examine a specific class of SPPDE that is characterised by two parameters of the form

$$L_{\varepsilon_1,\varepsilon_2}y \equiv \varepsilon_1 \frac{\partial^2 y}{\partial s^2} + \varepsilon_2 a(s)\frac{\partial y}{\partial s} - b(s)y - \frac{\partial y}{\partial t} = f(s,t) \tag{1}$$

where $(s,t) \in [0,1] \times [0,T]$, subject to

$y(s,0) = y_0(s),\ s \in (0,1), y(0,t) = \phi(0,t), t \in [0,T],\ y(1,t) = \psi(0,t), t \in [0,T]$(2)

with $0 < \varepsilon_1, \varepsilon_2 \leq 1$ and $a(s), b(s), f(s,t)$ are smooth enough and satisfy $b(s) \geq \beta > 0$ and $a(s) \geq \alpha > 0$ where $\alpha, \beta$are real numbers.We impose $y_0(0) = \phi(0,0)\ and\ y_0(1) = \psi(1,0)$, so that the data agrees at the extreme points (0,0) and (1,0). The conditions show that there is a constant C which is $\varepsilon_1, \varepsilon_2$ independent.

$$|y(s,t) - y(s,0)| = |y(s,t) - y_0(s)| \leq Ct$$

$$|y(s,t) - y(1,t)| = |y(s,t) - \psi(1,t)| \leq Cs. \tag{3}$$

Lemma 1.1 Maximum principle: Let $\varphi(s, t_k)$ be any function which is sufficiently smooth and satisfying $\varphi(0, t_k), \varphi(1, t_k) \geq 0$. Then

$L\varphi(s, t_k) \leq 0$ for $s \in [0,1] \Rightarrow \varphi(s, t_k) \geq 0$ for all $s \in [0,1]$.

Proof: The result is by contradictory. If possible, assume that there is a point $(s^*, t_k) \in [0, 1] \ni \varphi(s^*, t_k) < 0$ with $\varphi(s^*, t_k) = \min_{s \in [0,1]} \varphi(s, t_k)$.

It is clear that $\varphi_s(s^*, t_k) = 0, \varphi_t(s^*, t_k) = 0, \varphi_{ss}(s^*, t_k) \geq 0$.

Thus,

$$L\varphi(s^*, t_k) = \varepsilon_1 \varphi_{ss}(s^*, t_k) + \varepsilon_2 a(s) \varphi_s(s^*, t_k) - b(s)\varphi(s^*, t_k) - \varphi_t(s^*, t_k) > 0$$

which contradicts to given hypothesis that $L\varphi(s^*, t_k) \leq 0$ for all $s \in (0,1)$.

Therefore, it follows that $\varphi(s, t_k) \geq 0$ for all $s \in [0,1]$.

Lemma 1.2 The solution $y(s,t)$ of Eqs. (1), (2) is constrained by $|y(s,t)| \leq C$, for $(s,t) \in (0,1)$.

Proof. By Eq. (3), $|y(s, t)| \leq Ct, (s, t)\ in\ (0,1)$. Since $t \in [0, T]$, the solution is bounded, hence Ct is again constrained by another constant C. Thus $|y(s, t)| \leq C$, for $(s, t) \in (0, 1)$.

1.3 Discretization of the problem

1.3.1 Time discretization

Using the implicit Euler method, the time variable is discretised with a constant step size, denoted as τ, such that $[0, T]$ can be divided in to smaller intervals as $t_0 = 0, t_K = T, t_k = k\tau, k = 1,2, \ldots, K, \tau = T/K$. Using the above discretization, we can write the

Eq. (1) as

$$L_s z \equiv \varepsilon_1 z_{ss}(s,\ t_k) + \varepsilon_2 a(s) z_x(s,\ t_k) - \left(b(s) + \frac{1}{\tau}\right) z\,(s, t_k) = f(s,\ t_k) - \frac{1}{\tau} z(s,\ t_{k-1}) \quad (4)$$

with $z\,(\,s, 0) = y_0,\ \ s \in (0,1), z(0, t_k) = z(1, t_k) = 0$

The equation that defines the roots representing the solution of Eq. (1) is

$$\varepsilon_1 \lambda^2(s) + \varepsilon_2 a(s)\lambda(s) - \left(\frac{1}{\tau} + b(s)\right) = 0 \quad (5)$$

Two continuous functions produced by Eq. (5) are

$$\lambda_1(s) = -\frac{\varepsilon_2 a(s)}{2\varepsilon_1} - \sqrt{\frac{\left(b(s)+\frac{1}{\tau}\right)}{\varepsilon_1} + \left(\frac{\varepsilon_2 a(s)}{2\varepsilon_1}\right)^2} \quad (6)$$

$$\lambda_2(s) = -\frac{\varepsilon_2 a(s)}{2\varepsilon_1} + \sqrt{\frac{\left(b(s)+\frac{1}{\tau}\right)}{\varepsilon_1} + \left(\frac{\varepsilon_2 a(s)}{2\varepsilon_1}\right)^2} \quad (7)$$

These two real solutions characterize the boundary layers at $s = 0$ and $s = 1$.

Let $\theta_0 = -\max_{s\in[0,1]} \lambda_1(s)$ and $\theta_1 = \max_{s\in[0,1]} \lambda_2(s)$.

Utilizing the two fitting factors to effectively regulate the parameters in Eq. (4), we have

$$L_s z_j \equiv \varepsilon_1 \sigma_j z_{ss}(s_j,\ t_k) + \varepsilon_2 \eta_j a(s_j) z_s(s_j,\ t_k) - \left(b(s_j) + \frac{1}{\tau}\right) z(s_j,\ t_k) = f(s_j,\ t_k) - \frac{1}{\tau} z\,(s_j, t_{k-1}) \quad (8)$$

with $z(s_j, 0) = y_0, s_j \in (0,1), z(0, t_k\,) = z(1, t_k\,) = 0$

The error caused by local truncation of the scheme Eq. (8) is $e_k = z(\,s, t_k) - Z(s,\ t_k)$, where $Z(s,\ t_k)$ is the solution of Eq. (1), e_k is the truncation error in the time discretization at instant t_k.

Lemma 1.3 Suppose $\left|\frac{\partial^j z(s,t)}{\partial t^j}\right| \leq C$, for $(s,t) \in (0,1), j = 0, 1, 2$. Then measure of the discrepancy between the exact solution and the numerical approximation at a specific time step is $\|e_k\|_0 \leq C\tau^2$.

Proof. $z(s_j, t_k) - z(s_j, t_{k-1}) = \tau[\varepsilon_1 \sigma_j z_{ss}(s_j,\ t_k) + \varepsilon_2 \eta_j a(s_j) z_s(s_j,\ t_k) -$

$$b(s_j) z(s_j,\ t_k) - f(s_j,\ t_k) \tag{9}$$

Also $z(s, t_{k-1}) = z(s, t_k) - \tau z_t(s, t_k) + \int_{t_{k-1}}^{t_k} (t_{k-1} - \zeta) z_{tt}(s, \zeta) d\zeta.$ (10)

Using Eq. (8) in Eq. (10),

$$z(\ s_j, t_k) - z(s_j, t_{k-1}) = \tau[\varepsilon_1 \sigma_j z_{ss}(s_j,\ t_k) + \varepsilon_2 \eta_j a(s_j) z_s(s_j,\ t_k) -$$

$$b(s_j)\ z(\ s_j,\ t_k) -\ f(s_j,\ t_k)] + O(\tau^2) \tag{11}$$

Subtracting Eq. (11) from Eq. (9) gives

$$\tau L(e_k) = O(\tau), \qquad e_k(0) = e_k(1) = 0. \tag{12}$$

With this we can get the desired estimate using application of the Lemma 1.2 on the operator *L*.

1.3.2 Spatial discretization

Now, let *N* subintervals be assigned to the space interval [0, 1] such that $s_0 = 0, s_N$=1, $s_i = s_0 + ih, i = 1, \ldots, N$ where $h = s_i - s_{i-1}$.

The problem aims to find a numerical solution in the form of a trigonometric spline function for the j^{th} segment, denoted by $S_j(s)$, has the form

$$S_j(s) = \hat{a}_j + \hat{b}_j(s - s_j) + \hat{c}_j \sin v(s - s_j) + \hat{d}_j \cos v(s - s_j) \tag{13}$$

for $j = 0, 1, \ldots, N-1$, where $\hat{a}_j$, $\hat{b}_j$, $\hat{c}_j$ and $\hat{d}_j$ are constants and v is free parameter.

Let the exact solution be $z(s)$ and z_j approximate $z(s_j)$ which is obtained by trigonometric spline $S_j(s)$. The spline traverses through the points (s_j, z_j) and (s_{j+1}, z_{j+1}). The spline $S_j(s)$ fulfils the interpolation requirements at s_j and s_{j+1} as well as the first derivative continuity requirements at the common nodes (s_j, z_j). The trigonometric function $S(s)$ of class $C^2[a,b]$ interpolates $z(s)$ at the grid points s_j, for $j = 0,1,2,\ldots,N$, reliant upon a parameter v, and becomes a regular spline $S(s)$ in $[a, b]$ as $v \to 0$.

Let $S_j(s_j) = z_j, S_j(s_{j+1}) = z_{j+1}, \; S_j''(s_j) = M_j$ and $S_j''(s_{j+1}) = M_{j+1}$

Then by simple calculations, we get

$$\hat{a}_j = z_j + \frac{M_j}{v^2}, \hat{b}_j = \frac{z_{j+1} - z_j}{h} + \frac{M_{j+1} - M_j}{v\theta}, \hat{c}_j = \frac{M_j \cos\theta - M_{j+1}}{v^2 \sin\theta}, \hat{d}_j = -\frac{M_j}{v^2}$$

where $\theta = vh$, for $j = 0,1,\ldots,N$.

Using the condition $S_j'(s_j^+) = S_j'(s_j^-)$, we get the following relation

$$rM_{j+1} + 2\rho M_j + rM_{j-1} = \frac{1}{h^2}(z_{j+1} - 2z_j + z_{j-1}) \text{ for } j = 1,2,3,\ldots,N-1. \tag{14}$$

here $r = \frac{-1}{\theta^2} + \frac{1}{\theta \sin\theta}, \rho = \frac{1}{\theta^2} - \frac{\cos\theta}{\theta \sin\theta}, M_j = z_{ss}(s_j, t_k)$

Rearranging the Eq. (8),

$$\varepsilon_1 \sigma_j z_{ss}(s_j, t_k) = \hat{p}(s_j) z_s(s_j,\ t_k) + \hat{q}(\ s_j) z(\ s_j, t_k) + f(s_j,\ t_k) - \frac{1}{\tau} z(\ s_j,\ t_{k-1})$$

where $\hat{p}(s) = -\varepsilon_2 \eta(s) a(s), \hat{q}(s) = b(s) + \frac{1}{\tau}$

By using second derivatives of spline, we have

$$\varepsilon_1 \sigma_j M_j = \hat{p}(s_j) z_s(s_j,\ t_k) + \hat{q}(s_j) z(\ s_j,\ t_k) + f(s_j, t_k) - \frac{1}{\tau} z(\ s_j, t_{k-1}) \quad (15)$$

Let $z_s(s_{j+1}, t_k) \cong \frac{z(s_{j-1},t_k) - 4z(s_j,t_k) + 3z(s_{j+1},t_k)}{2h}$ (16)

$$z_s(s_{j-1}, t_k) \cong \frac{-3z(s_{j-1},t_k) + 4z(s_j,t_k) - z(s_{j+1},t_k)}{2h} \quad (17)$$

$$z_s(s_j\ , t_k) \cong \left(\frac{1 + 2\omega h^2 \hat{q}_{j+1} + \omega h(3\hat{p}_{j+1} + \hat{p}_{j-1})}{2h}\right) z(s_{j+1}\ ,\ t_k)$$

$$- 2\omega\ [\hat{p}_{j+1} + \ \hat{p}_{j-1}] z(s_j\ ,\qquad t_k)$$

$$- \left(\frac{1 + 2\omega h^2 \hat{q}_{j-1} - \omega h[\hat{p}_{j+1} + 3\hat{p}_{j-1}]}{2h}\right) z(\ s_{j-1}\ , t_k)$$

$$+ \omega\ h \left[f(\ s_{j+1}\ ,\ t_k) - \frac{1}{\tau} z(s_{j+1}\ ,\ t_{k-1}) - f(s_{j-1}\ ,\ t_k) + \frac{1}{\tau} z(s_{j-1}\ , t_{k-1})\right] \quad (18)$$

Using the Eqs. (15) - (18) in Eq. (14), we obtain the following tri-diagonal system $E_j z(s_{j-1}\ ,\ t_k) + F_j\ z(s_j\ , t_k) + G_j z(s_{j+1}\ ,\ t_k) = H_{j,k}$ for $j = 1,2,\dots, N-1$. (19)

Here

$$E_j = -\ \varepsilon_1 \sigma_j - \frac{3}{2} \mathrm{r}\ \hat{p}_{j-1}\ h\ + \rho\ \hat{p}_j h^2 \omega[\hat{p}_{j+1} + 3\hat{p}_{j-1}] - 2\omega\ \hat{p}_j \rho\ h^3 \hat{q}_{j-1} + \frac{r}{2} \hat{p}_{j+1} h$$

$$+ \qquad r\ \hat{q}_{j-1} h^2 - h\rho\ \hat{p}_j$$

$$F_j = 2\varepsilon_1\sigma_j + 2r\,\hat{p}_{j-1}\,h - 4\rho\,\hat{p}_j h^2\omega\left[\hat{p}_{j+1} + \hat{p}_{j-1}\right] - 2r\,\hat{p}_{j+1}h + 2\rho\,\hat{q}_j h^2$$

$$G_j = -\varepsilon_1\sigma_j - \frac{r}{2}\hat{p}_{j-1}\,h + \rho\,\hat{p}_j h^2\omega\left[3\hat{p}_{j+1} + \hat{p}_{j-1}\right] + 2\omega h^3\rho\,\hat{p}_j\hat{q}_{j+1} + \frac{3}{2}\mathrm{r}\,\hat{p}_{j+1}h + r\,\hat{q}_{j+1}h^2 + h\rho\,\hat{p}_j$$

$$H_{j,k} = -h^2\left[\left(r - 2\omega\rho\,\hat{p}_j h\right)\left(f_{j-1,k} - \frac{1}{\tau}z_{j-1,k-1}\right) + 2\rho\left(f_{j,k} - \frac{1}{\tau}z_{j,k-1}\right) + \left(r + 2\omega\rho\,\hat{p}_j h\right)\left(f_{j+1,k} - \frac{1}{\tau}z_{j+1,k-1}\right)\right]$$

Fitting factors can be acquired using Eq. (6) and Eq. (7) in (19) as

$$\sigma_j = \frac{-\left(b(s_j) + \frac{1}{\tau}\right)\beta h}{4}\left(\frac{e^{\left(\frac{-\varepsilon_2 a(s_j)h}{2\varepsilon_1}\right)}}{sinh\left(\frac{\lambda_1(s_j)h}{2}\right)sinh\left(\frac{\lambda_2(s_j)h}{2}\right)}\right) \quad \text{for } j = 1,2,\ldots,N$$

$$\eta_j = \frac{\left(b(s_j)+\frac{1}{\tau}\right)h}{2\varepsilon_2 a(s_j)}\left(coth\left(\frac{\lambda_1(s_j)h}{2}\right) + coth\left(\frac{\lambda_2(s_j)h}{2}\right)\right) \quad \text{for } j = 1,2,\ldots,N \text{ where } \beta = \frac{h}{\varepsilon_1}.$$

1.4Convergence Analysis

This new method has a local truncation error of

$$T_j(h) = [-1 + 2(\alpha+\beta)]h^2\varepsilon_1 z_j'' + \left\{\left[\left(4\omega\varepsilon_1 + \frac{1}{3}\right)\beta - \frac{2\alpha}{3}\right]\hat{p}_j z_j''' + (-1 + 12\alpha)\frac{\varepsilon_1}{2}z_j^{iv}\right\}h^4 + O(h^6)$$

Then for $\alpha = \frac{1}{12}, \beta = \frac{5}{12}, \omega = \frac{-1}{20\varepsilon_1}, T(h) = O(h^6)$.

with boundary conditions, matrix form of Eq. (19) can be

$$(A+B)Z+\hat{P}+T(h)=O \qquad (20)$$

where

$$A=\begin{bmatrix} 2\varepsilon_1\sigma_1 & -\varepsilon_1\sigma_1 & 0 & 0 & \dots & 0 \\ -\varepsilon_1\sigma_1 & 2\varepsilon_1\sigma_1 & 2\varepsilon_1\sigma_1 & 0 & \dots & 0 \\ 0 & -\varepsilon_1\sigma_1 & 2\varepsilon_1\sigma_1 & -\varepsilon_1\sigma_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -\varepsilon_1\sigma_1 & 2\varepsilon_1\sigma_1 \end{bmatrix}$$

$$B=\begin{bmatrix} u_1v_1 & 0 & 0 & \dots & 0 \\ r_2u_2v_2 & 0 & \dots & 0 \\ 0 & r_3u_3v_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & r_{N-1} & u_{N-1} \end{bmatrix}$$

$$r_j=-\frac{3}{2}\mathrm{r}\,\hat{p}_{j-1}\,h\;+\rho\,\hat{p}_jh^2\omega\left[\hat{p}_{j+1}+3\hat{p}_{j-1}\right]-2\omega\,\hat{p}_j\rho\,h^3\hat{q}_{j-1}+\frac{r}{2}\hat{p}_{j+1}h+r\,\hat{q}_{j-1}h^2$$
$$-h\rho\,\hat{p}_j$$

$$u_j=2r\,\hat{p}_{j-1}\,h-4\rho\,\hat{p}_jh^2\omega\left[\hat{p}_{j+1}+\;\hat{p}_{j-1}\right]-2r\,\hat{p}_{j+1}h+2\rho\,\hat{q}_jh^2$$

$$v_j=-\frac{r}{2}\hat{p}_{j-1}\,h\;+\rho\,\hat{p}_jh^2\omega\left[3\hat{p}_{j+1}+\hat{p}_{j-1}\right]+2\omega h^3\rho\,\hat{p}_j\hat{q}_{j+1}+\frac{3}{2}\mathrm{r}\,\hat{p}_{j+1}h+r\,\hat{q}_{j+1}h^2+$$
$h\rho\,\hat{p}_j$ for $j=1,2,\dots,N-1$

$$\hat{P}=[\hat{q}_1+(-\varepsilon_1\sigma_1+r_1)\gamma_0,\hat{q}_2,\hat{q}_3,\dots,\hat{q}_{N-1}+(-\varepsilon_1\sigma_1+v_1)\gamma_1$$

$$\hat{q}_j=-h^2\Big[(r-2\omega\rho\,\hat{p}_jh)(f_{j-1,k}-\frac{1}{\tau}z_{j-1,k-1})+2\rho(f_{j,k}-\frac{1}{\tau}z_{j,\;k-1})$$
$$+\left(r+2\omega\rho\,\hat{p}_jh\right)(f_{j+1,\;k}-\frac{1}{\tau}z_{j+1,\;k-1})\Big],\;\; j=1,2,3,\dots,N-1.$$

$$Z=\left[z_1^k,z_2^k,\dots,z_{N-1}^k\right]^T,T(h)=[T_1,T_2,T_3,\dots,T_{N-1}]^T$$

Let $Y = [y_1^k, y_2^k, \ldots, y_{N-1}^k]^T$ be the solution of Eqs. (1) - (2), we have

$$(A + B)Y + \hat{P} = O \qquad (21)$$

Let $e_j^k = y_j^k - z_j^k, j = 1,2, \ldots, N-1$ be the error of discretization and $E = [e_1^k, e_2^k, \ldots, e_{N-1}^k]^T$

Using Eq. (20) and Eq. (21), we get $(A + B)E = T(h)$ (22)

Let $\tilde{L}_j$ be the sum of the elements of j^{th} row of *(A+B)*, then

$$\tilde{L}_j = \varepsilon_1 \sigma_1 + \frac{r}{2}(3\hat{p}_{j-1} - \hat{p}_{j+1})h - h\rho\, \hat{p}_j + (r\, \hat{q}_{j+1} + 2\rho \hat{q}_j)h^2 - \rho\, \hat{p}_j h^2 \omega[\hat{p}_{j+1} + 3\hat{p}_{j-1}] + 2\omega\, \hat{p}_j \rho\, h^3 \hat{q}_{j+1} \quad \text{for } j = 1$$

$$\tilde{L}_j = h^2(\mathrm{r}\hat{q}_{j-1} + 2\mathrm{\rho}\hat{q}_j + \mathrm{r}\hat{q}_{j+1}) + 2h^3 \mathrm{\rho\omega} \hat{p}_j (\hat{q}_{j+1} - \hat{q}_{j-1}), \qquad j = 2, \ldots, N-2$$

$$\tilde{L}_j = \varepsilon_1 \sigma_1 + \frac{r}{2}(\hat{p}_{j-1} - 3\hat{p}_{j+1})h - h\rho\, \hat{p}_j + (r\, \hat{q}_{j-1} + 2\rho \hat{q}_j)h^2 - \rho\, \hat{p}_j h^2 \omega[\hat{p}_{j+1} + \hat{p}_{j-1}] - 2\omega\, \hat{p}_j \rho\, h^3 \hat{q}_{j-1} \quad \text{for } j = N-1$$

Let $W_{1^*} = \min_{1 \le j \le N} |\tilde{p}(s)|,\ W_1^* = \max_{1 \le j \le N} |\tilde{p}(s)|,$ and $W_{2^*} = \min_{1 \le j \le N} |\tilde{q}(s)|,\ W_2^* = \max_{1 \le j \le N} |\tilde{q}(s)|.$

Since $0 < \varepsilon_1 << 1$ and $\varepsilon_1 \propto O(h)$,it is clear that for$h$, $(A + B)$ is monotone.

As $(A + B)^{-1}$ exists and $(A + B)^{-1} \ge 0$, from Eq. (22), we have

$$\|E\| \le \|(A + B)^{-1}\|.\|T\|. \qquad (23)$$

Let $(A + B)_{j,k}^{-1}$ be the $(j,\ k)^{th}$ element of $(A + B)^{-1}$ and

$$\|(A + B)^{-1}\| = \max_{1 \le j \le N-1} \sum_{k=1}^{N-1} (A + B)_{j,k}^{-1} \quad \text{and} \quad \|T(h)\| = \max_{1 \le j \le N-1} |T_j(h)| \qquad (24)$$

Since $(A+B)^{-1}_{j,k} \geq 0$ and $\sum_{k=1}^{N-1}(A+B)^{-1}_{j,k}.\tilde{L}_k = 1,\ j = 1,2,3,\ldots,N-1.$

$$(A+B)^{-1}_{j,k} \leq \frac{1}{L_j} < \frac{1}{h^2[(r+2\rho)W_{2^*} - 4\rho\omega W_{1^*}^2]}, j = 1,\ldots,\mathrm{N}-1 \quad (25)$$

Furthermore, $\sum_{k=1}^{N-1}(A+B)^{-1}_{j,k} \leq \frac{1}{\min\limits_{2\leq j\leq N-2} L_j} < \frac{1}{h^2(2(r+\rho)W_{2^*})}.$ (26)

Using Eqs. (25) - (26) in Eq. (23), we get $\|E\| \leq O(h^4).$ (27)

Thus, from Eq. (12) and Eq. (27), we obtain

$$\max_{0\leq j\leq N, 0\leq k\leq K} \left|y_j^k - z_j^k\right| \leq C(h^4 + \tau) \quad (28)$$

which demonstrates that our method exhibits fourth-order convergence in space and first order in time for $\alpha = \frac{1}{12}, \beta = \frac{5}{12}, \omega = -\frac{1}{20\varepsilon_1}$ on uniform mesh.

1.5Numerical Results

Conducted In order to computationally demonstrate the proposed method, we examine a class of SPPDE that consists of two parameters. For each mesh node, we determine the maximum absolute error (MAE) by $E_{\varepsilon_1,\varepsilon_2}^{N,K} = \max\limits_{0\leq j\leq N; 0\leq k\leq K} \left|\left(y_{\varepsilon_1,\varepsilon_2}^{N,K}\right)_{j,k} - \left(y_{\varepsilon_1,\varepsilon_2}^{2N,2K}\right)_{j,k}\right|$ when exact solution is unknown.

Example 1.1 $\varepsilon_1 \frac{\partial^2 y}{\partial s^2} + \varepsilon_2(1+s)\frac{\partial y}{\partial s} - y(s) - \frac{\partial y}{\partial t} = 16s^2(1-s)^2,$

where $y(s,0) = 0 \forall s \in (0,1); y(0,t) = 0, y(1,t) = 0$ for $t \in [0,1]$.

The results obtained from Tables 1.1 and 1.2 are compared with the corresponding values in Tables 1 and 5 of [15] for the different parameter $\varepsilon_1$and ε_2. The result profile at the specified nodes is shown in Fig. 1.1 for Example 1.1.

Example 1.2 $\varepsilon_1 \frac{\partial^2 y}{\partial s^2} - (2 - s^2)\frac{\partial y}{\partial s} - sy(s) - \frac{\partial y}{\partial t} = -10t^2 e^{-t} s(1 - s)$

where $y(s, 0) = 0, \forall s \in (0,1), y(0, t) = 0, y(1, t) = 0$, for $t \in (0,3)$.

Table 1.3 shows the MAEs for range of ε_1 and fixed ε_2 in comparison with the results of [18]. The result profile at the specified nodes is shown in Fig. 1.2 for Example 1.2.

Example 1.3

$$\varepsilon_1 \frac{\partial^2 y}{\partial s^2} + \varepsilon_2(1 + s(1 - s) + t^2)\frac{\partial y}{\partial s} - \frac{\partial y}{\partial t} - (1 + 5st)y = s(1 - s)(e^t - 1)$$

where $y(s, 0) = 0 \ \forall s \in (0,1), y(0, t) = 0, y(1, t) = 0$, for $t \in [0,1]$.

The comparison of the MAEs is shown in Tables 1.4 for range of ε_1 and fixed ε_2 with the results of [4]. The result profile at the specified nodes is shown in Fig. 1.3 for Example 1.3.

Table 1.1 Comparison of MAEs with $\varepsilon_2 = 2^{-2}$ and *K(=N)* for Example 1.1.

$\varepsilon_1 \backslash N$	8	16	32	64	128	256	512
Our results							
2^{-2}	1.15E-2	2.92E-3	7.39E-4	1.86E-4	4.68E-5	1.17E-5	2.93E-6
2^{-4}	3.46E-2	1.02E-2	2.78E-3	7.24E-4	1.84E-4	4.66E-5	1.17E-5
2^{-6}	6.25E-2	2.63E-2	8.85E-3	2.58E-3	6.97E-4	1.81E-4	4.61E-5
2^{-8}	5.64E-2	3.23E-2	1.67E-2	6.76E-3	2.23E-3	6.47E-4	1.74E-4
2^{-10}	5.49E-2	1.85E-2	5.60E-4	1.20E-2	8.08E-3	4.23E-3	1.70E-3
2^{-12}	5.91E-2	1.50E-2	5.89E-3	3.71E-3	2.91E-3	2.02E-3	1.06E-3
2^{-14}	6.29E-2	1.98E-2	6.12E-3	2.55E-3	1.28E-3	8.91E-4	7.23E-4
2^{-38}	6.48E-2	2.34E-2	9.41E-3	3.76E-3	1.60E-3	7.33E-4	3.48E-4

2^{-40}	6.48E-2	2.34E-2	9.41E-3	3.76E-3	1.60E-3	7.33E-4	3.48E-4
Results in [15]							
2^{-2}	0.17E-1	0.71E-2	0.31E-2	0.14E-2	0.68E-3	0.33E-3	0.16E-3
2^{-4}	0.30E-1	0.11E-1	0.45E-2	0.19E-2	0.86E-3	0.41E-3	0.20E-3
2^{-6}	0.51E-1	0.19E-1	0.72E-2	0.29E-2	0.12E-2	0.58E-3	0.28E-3
2^{-8}	0.62E-1	0.32E-1	0.13E-1	0.49E-2	0.19E-2	0.81E-3	0.37E-3
2^{-10}	0.62E-1	0.34E-1	0.17E-1	0.82E-2	0.32E-2	0.12E-2	0.49E-3
2^{-12}	0.62E-1	0.34E-1	0.17E-1	0.88E-2	0.44E-2	0.20E-2	0.81E-3
2^{-14}	0.62E-1	0.34E-1	0.17E-1	0.88E-2	0.44E-2	0.22E-2	0.11E-3
2^{-40}	0.62E-1	0.34E-1	0.17E-1	0.88E-2	0.44E-2	0.22E-2	0.11E-3

Table1.2 Comparison of MAEs with $\varepsilon_1 = 2^{-5}$and *N (=2K)* for Example 1

$\varepsilon_2 \backslash N$	2^5	2^6	2^7	2^8	2^9	2^{10}
Our results						
2^{-2}	0.549E-2	0.144E-2	0.369E-3	0.931E-4	0.234E-4	0.586E-5
2^{-4}	0.557E-2	0.144E-2	0.369E-3	0.931E-4	0.234E-4	0.586E-5
2^{-6}	0.557E-2	0.144E-2	0.369E-3	0.931E-4	0.234E-4	0.586E-5
2^{-8}	0.557E-2	0.144E-2	0.369E-3	0.931E-4	0.234E-4	0.586E-5
2^{-40}	0.557E-2	0.144E-2	0.369E-3	0.931E-4	0.234E-4	0.586E-5
Results in [15]						
2^{-2}	0.086E-1	0.039E-1	0.018E-1	0.092E-2	0.045E-2	0.022E-2
2^{-4}	0.075E-1	0.032E-1	0.015E-1	0.073E-2	0.036E-2	0.017E-2
2^{-6}	0.074E-1	0.032E-1	0.014E-1	0.071E-2	0.035E-2	0.017E-2
2^{-8}	0.074E-1	0.032E-1	0.014E-1	0.071E-2	0.035E-2	0.017E-2

2^{-40}	0.074E-1	0.032E-1	0.014E-1	0.071E-2	0.035E-2	0.017E-2

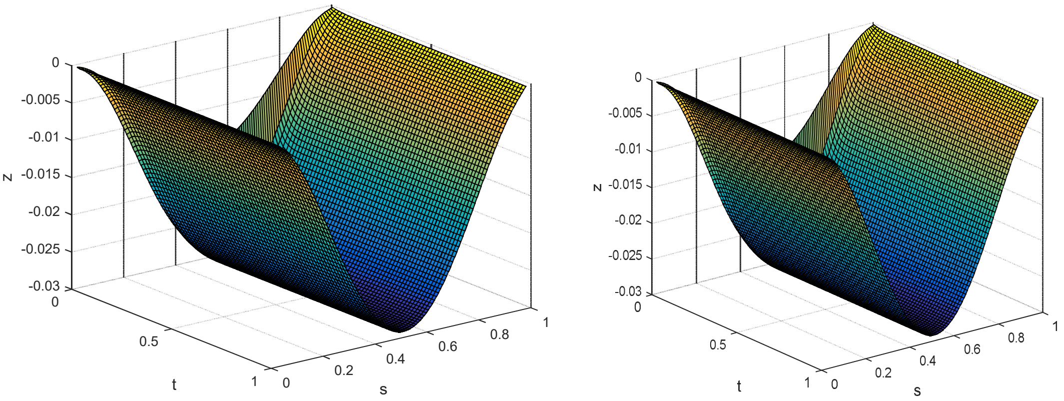
TABLE1.3 Similitude of MAEs for $\varepsilon_2 = 2^{-0}$ and N in Example 1.2

$\varepsilon_1 \downarrow$ $N \rightarrow$	2^4	2^5	2^6	2^7	2^8
$\tau \rightarrow$	1/10	1/20	1/40	1/80	1/160
Our Results					
2^{-8}	9.1680E-3	2.6628E-3	2.3183E-3	1.5931E-3	6.8094E-4
2^{-10}	1.1708E-2	2.9057E-3	9.0329E-4	6.4602E-4	5.8001E-4
2^{-12}	1.6118E-2	4.2474E-3	9.6329E-4	4.0205E-4	2.2189E-4
2^{-14}	1.7922E-2	5.0965E-3	1.2884E-3	3.9533E-4	1.9563E-4
2^{-16}	1.8449E-2	5.3770E-3	1.4355E-3	4.5694E-4	1.9543E-4
Results in [18]					
2^{-8}	0.7863E-2	0.4366E-2	0.2202E-2	0.8796E-3	0.2670E-3
2^{-10}	0.7863E-2	0.4370E-2	0.2295E-2	0.1174E-2	0.5706E-3
2^{-12}	0.7863E-2	0.4370E-2	0.2295E-2	0.1175E-2	0.5944E-3
2^{-14}	0.7863E-2	0.4370E-2	0.2295E-2	0.1175E-2	0.5944E-3
2^{-16}	0.7863E-2	0.4370E-2	0.2295E-2	0.1175E-2	0.5944E-3

TABLE1.4 Similitude of MAEs for $\varepsilon_2 = 10^{-7}$ in Example 1.3

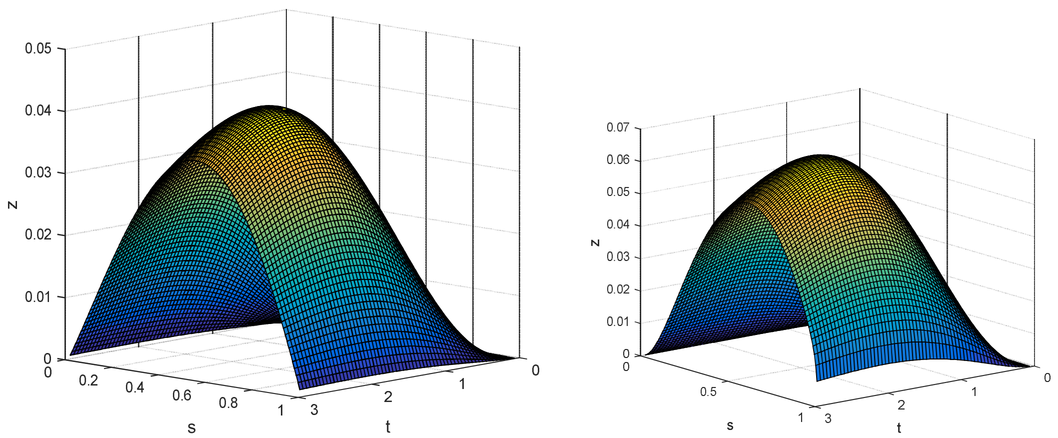
$N \rightarrow$	64	128	256	512
$\varepsilon_1 \downarrow \tau \rightarrow$	$1/2^4$	$1/2^5$	$1/2^6$	$1/2^7$
Our Results				
10^{-6}	7.0053E-4	4.5344E-5	1.1392E-5	1.7946E-4
10^{-7}	7.0036E-4	4.5341E-5	1.1391E-5	1.7944E-4
10^{-8}	7.0895E-4	4.5490E-5	1.1410E-5	1.8059E-4
10^{-9}	8.4775E-4	4.7637E-5	1.1678E-5	1.9785E-4

Results in [4]				
10^{-6}	0.96949E-3	0.25231E-3	0.49906E-3	0.12824E-3
10^{-7}	0.98712E-3	0.25485E-3	0.50049E-3	0.12853E-3
10^{-8}	0.95128E-3	0.25237E-3	0.50026E-3	0.12781E-3
10^{-9}	0.96746E-3	0.25237E-3	0.50012E-3	0.12803E-3



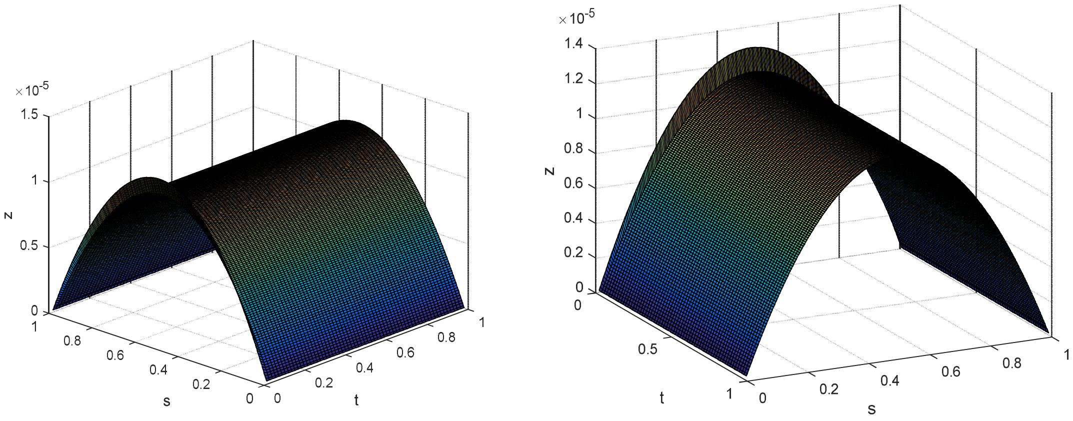
$\varepsilon_1 = 2^{-8}, \varepsilon_2 = 2^{-2}\varepsilon_1 = 2^{-5}, \varepsilon_2 = 2^{-8}$

Figure 1.1 Result profile for Example 1.1 for diverse values of ε_1 and ε_2 with $K = 64, N = 128$



$\varepsilon_1 = 2^{-2}, \varepsilon_2 = 2^{-6}$ $\varepsilon_1 = 2^{-8}, \varepsilon_2 = 2^{-0}$

Figure 1.2 Result profile for Example 1.2 for diverse values of ε_1 and ε_2 with $N = 128\,, K = 64$



$\varepsilon_1 = 10^{-6}, \varepsilon_2 = 10^{-7} \varepsilon_1 = 10^{-7}, \varepsilon_2 = 10^{-6}$

Figure 1.3 Result profile for Example 1.3 for diverse values of ε_1 and ε_2 with $K = 32, N = 128$.

1.6Conclusion

The desire to enhance accuracy and convergence in solving SPPDEs drives our efforts to develop a uniformly convergent numerical scheme of higher order. We proposed a scheme to examine the solution of a SPPDE of diffusion-convection type with two small parameters. We partially discretized the continuous problem in the temporal direction using the implicit Euler scheme. Use of the trigonometric spline method to discretize space on a constant mesh led to a fourth order convergence in space. Theoretical analysis to obtain the stability and error estimate shown that the proposed method is unconditionally stable. The convergence analysis has demonstrated that the method exhibits parameter uniformity. The accuracy of the scheme is determined by comparing it with some existing methods available in literature. For a given value of ε_2, we find that the MAEs in Table 1.1,1.3 and 1.4 remains unchanged as $\varepsilon_1$approaches to zero. It demonstrates that the method in use is up to a level of precision of ε_1. For fixed ε_1, h and $\tau,$ Table 1.2 also confirms the ε_2- uniform convergence.

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