Enhanced Lagrange's interpolation technique using an empirical model; A comparative study

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Abstract: Lagrange's method was developed to create polynomial functions passing through a given set of points, essential for numerical analysis and approximation theory. Lagrange's interpolation is widely used in engineering, medicine, business, and data analysis. This article introduces a modified interpolation technique using an empirical function, demonstrating improved results. The article includes a comparative study with standard examples and discusses error analysis.

Keywords: Lagrange's Interpolation; Polynomial; Approximation; Errors; Empirical model.

Introduction:

Joseph-Louis Lagrange, a well-known French mathematician of the 18th century (1736-1813), made significant contributions to several domains, one of the most influential being the formulation of the Lagrange interpolation formula. This formula offered a fresh and constructive approach to the problem of polynomial interpolation, which is a foundation of numerical analysis and approximation theory. Prior to Lagrange's work, determining a polynomial that accurately traveled through a given set of data points was a difficult and frequently computationally impossible process. Lagrange's method provided a systematic and elegant solution, paving the door for considerable advances in future numerical techniques ([1] – [4]).

Lagrange interpolation's central idea is that it may create a polynomial function that perfectly matches a certain set of points, known as nodes. This is accomplished by constructing a weighted sum of Lagrange basis polynomials, each of which is set to one at one node and zero at all others. By carefully combining these base polynomials, the final function seamlessly interpolates the given data, resulting in a continuous representation of the underlying relationship ([5] – [9]).

The influence of Lagrange's interpolation formula extends far beyond its first formulation. It laid the groundwork for many following advances in numerical methods, such as techniques for numerical integration, differentiation, and differential equation solution. Furthermore, Lagrange interpolation has had a significant impact on polynomial theory by offering a realistic approach for studying polynomial functions' features and behavior. Its impact can be observed in a wide range of applications, including curve fitting and data smoothing, computer graphics, and scientific visualization.

The formula's relative simplicity and conceptual clarity have made it a popular and enduring tool for mathematicians, physicists, and engineers alike, cementing Lagrange's reputation as a pioneer in numerical analysis. The capacity to generate polynomial functions that correspond to a given dataset remains critical for modern computational approaches ([10] – [14]).

From figure (i) $f(x)$ = exact function of which only $N+1$ discrete values are known and used to establish an interpolating or approximating function $g(x)$. This function will pass through all specified interpolation points (also referred to as data points or nodes).

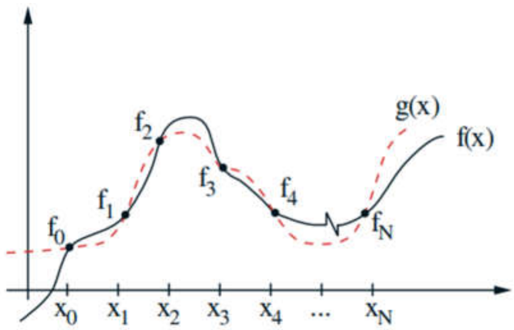

figure (i)

$$f(x_0) = f_0$$

$$f(x_1) = f_1$$

$$f(x_2) = f_2$$

$$\vdots$$

$$f(x_N) = f_N$$

Let $g(x) = \sum_{i=0}^{N} f_i \, L_i(x)$

Where $L_i(x)$ is a polynomial of degree N associated with each node i such that

$$L_i(x_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

For example, if we have 4 interpolation points

$$g(x_1) = f_0 \, L_0(x_1) + f_1 \, L_1(x_3) + f_2 \, L_2(x_3) + f_3 \, L_3(x_3)$$

By the definition of $L_i(x_j)$, $L_0(x_3) = 0$, $L_1(x_1) = 1$, $L_2(x_1) = 0$, $L_3(x_1) = 0$

$$\Longrightarrow g(x_1) = f_1$$

For N^{th} degree polynomial $L_i(x)$ satisfies the following conditions

i) It has the roots at $x_0, x_1 \; \dots \; x_{i-1}, x_{i+1} \dots x_N$

ii) $L_i(x_i) = 1$

The general form of the interpolating function $g(x)$ is

$$g(x) = \sum_{i=0}^{N} f_i\, L_i(x)$$

$$L_i(x) = \frac{(x-x_0),(x-x_1)\ \dots\ (x-x_{i-1}),(x-x_{i+1})\dots(x-x_N)}{(x_i-x_0),(x_i-x_2)\ \dots\ (x_i-x_{i-1}),(x_i-x_{i+1})\dots(x_i-x_N)}$$

Errors associated with LaGrange's Interpolation:

Using Taylor's series analysis, error is given by

$$E(x) = f(x) - g(x)$$

$$E(x) = L(x) f^{(n+1)}(\xi) \qquad , \quad x_0 \le \xi \le x_N$$

Where $L(x) = \frac{(x-x_0),(x-x_1)\ \dots.(x-x_N)}{(N+1)!}$, $f^{(N+1)}(\xi)$ is $N+1^{th}$ derivative of f

With respect to x at ξ.

Enhanced method of LaGrange's interpolation:

Given function values at $N+1$ nodal values are

$$f(x_0) = f_0$$
$$f(x_1) = f_1$$
$$f(x_2) = f_2$$
$$\vdots$$
$$f(x_N) = f_N$$

Then we can write new empirical function

$$h(x) = \sum_{i=0}^{N} f_i\ R_i(x) \quad \text{where} \quad R_i(x) = \frac{L_i(x)}{\cosh\left(\frac{x}{x_i}-1\right)}$$

here $L_i(x) = \frac{(x-x_0),(x-x_1)\ \dots(x-x_{i-1}),(x-x_{i+1})\dots(x-x_N)}{(x_i-x_0),(x_i-x_2)\ \dots(x_i-x_{i-1}),(x_i-x_{i+1})\dots(x_i-x_N)}$

$$R_i(x_j) = \begin{cases} 0 & i \ne j \\ 1 & i = j \end{cases}$$

Error $E(x) = f(x) - h(x)$

$$E(x) = R(x) f^{(n+1)}(\xi) \qquad , \quad x_0 \le \xi \le x_N$$

$$R(x) = \frac{(x-x_0),(x-x_1)\ \dots(x-x_N)}{(N+1)!\ \max\limits_{x_0 \le x \le x_N} \cosh\left(\frac{x}{\xi}-1\right)}$$

Eample1: Consider the following table values find $g(0.5)$ by Lagrange's method. Estimate the error.

i	x_i	$f(x_i) = \ln(x_i)$
0	0.3	-1.203973
1	0.4	-0.916291
2	0.6	-0.510825
3	0.7	-0.356675

Sol:

$$g(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f_3$$

$$g(0.5) = -\frac{(0.5-0.4)(0.5-0.6)(0.5-0.7)}{(0.3-0.4)(0.3-0.6)(0.3-0.7)} \times 1.203973 - \frac{(0.5-0.3)(0.5-0.6)(0.5-0.7)}{(0.4-0.3)(0.4-0.6)(0.4-0.7)} \times 0.916291 - \frac{(0.5-0.3)(0.5-0.4)(x-0.7)}{(0.6-0.3)(0.6-0.4)(0.6-0.7)} \times 0.510825 - \frac{(0.5-0.3)(0.5-0.4)(x-0.6)}{(0.7-0.3)(0.7-0.4)(0.7-0.6)} \times 0.356675$$

$g(0.5) = -0.6913026$

$e(x) = L(x) f^{(4)}(\xi) \quad , \quad 0.3 \le \xi \le 0.7$

$e(0.5) = \frac{(0.5-0.3)(0.5-0.4)(0.5-0.6)(0.5-0.7)}{(3+1)!} f^{(4)}(0.5)$

$f(x) = \ln x \, , f^{(1)} = x^{-1} \; , f^{(2)} = -x^{-2} \; , f^{(3)} = 2x^{-3} \; , f^{(4)} = -6x^{-4}$

$f^{(4)}(0.5) = -96$

$e(0.5) = \frac{(0.5-0.3)(0.5-0.4)(0.5-0.6)(0.5-0.7)}{(4)!} \times 96 = 0.0016$

Exact error is

$$f(0.5) - g(0.5) = -0.69314718 + 0.691303 = -0.0018$$

Enhanced Lagrange's interpolation:

$$h(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \frac{f_0}{\cosh\left(\frac{x_0}{x}-1\right)} + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \frac{f_1}{\cosh\left(\frac{x_1}{x}-1\right)} + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \frac{f_2}{\cosh\left(\frac{x_2}{x}-1\right)} + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \frac{f_3}{\cosh\left(\frac{x_3}{x}-1\right)}$$

$$h(0.5) = -\frac{(0.5-0.4)(0.5-0.6)(0.5-0.7)}{(0.3-0.4)(0.3-0.6)(0.3-0.7)} \times 1.11368 - \frac{(0.5-0.3)(0.5-0.6)(0.5-0.7)}{(0.4-0.3)(0.4-0.6)(0.4-0.7)} \times 0.898266 - \frac{(0.5-0.3)(0.5-0.4)(x-0.7)}{(0.6-0.3)(0.6-0.4)(0.6-0.7)} \times 0.500776 - \frac{(0.5-0.3)(0.5-0.4)(x-0.6)}{(0.7-0.3)(0.7-0.4)(0.7-0.6)} \times 0.329927$$

$h(0.5) = -0.691628$

$e(x) = L(x)\frac{f^{(4)}(\xi)}{\cosh\left(\frac{x}{\xi}-1\right)}$, $0.3 \leq \xi \leq 0.7$

$e(0.5) = \frac{(0.5-0.3)(0.5-0.4)(0.5-0.6)(0.5-0.7)}{(3+1)!\max\limits_{0.3\leq x\leq 0.7}\cosh\left(\frac{x}{0.5}-1\right)} f^{(4)}(0.5)$

$f(x) = \ln x$, $f^{(1)}(x) = x^{-1}$, $f^{(2)}(x) = -x^{-2}$, $f^{(3)}(x) = 2x^{-3}$, $f^{(4)}(x) = -6x^{-4}$

$f^{(4)}(0.5) = -96$

$e(0.5) = \frac{(0.5-0.3)(0.5-0.4)(0.5-0.6)(0.5-0.7)}{(4)!(1.23058)} \times -96 = -0.0013$

Exact error is

$$f(0.5) - g(0.5) = -0.69314718 + 0.691628 = -0.0015$$

Eample 2: Consider the following table values find $g(0.5)$ by Lagrange's method. Estimate the error.

i	x_i	$f(x_i) = \sin(x_i)$
0	$\frac{\pi}{6}$	0.5
1	$\frac{\pi}{3}$	0.866025
2	$\frac{\pi}{2}$	1

Sol:

$$g(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2$$

$$g\left(\frac{\pi}{4}\right) = \frac{\left(\frac{\pi}{4}-\frac{\pi}{3}\right)\left(\frac{\pi}{4}-\frac{\pi}{2}\right)}{\left(\frac{\pi}{6}-\frac{\pi}{3}\right)\left(\frac{\pi}{6}-\frac{\pi}{2}\right)} \times 0.5 + \frac{\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\left(\frac{\pi}{4}-\frac{\pi}{2}\right)}{\left(\frac{\pi}{3}-\frac{\pi}{6}\right)\left(\frac{\pi}{3}-\frac{\pi}{2}\right)} \times 0.866025 + \frac{\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\left(\frac{\pi}{4}-\frac{\pi}{3}\right)}{\left(\frac{\pi}{2}-\frac{\pi}{6}\right)\left(\frac{\pi}{2}-\frac{\pi}{3}\right)} \times 1$$

$g\left(\frac{\pi}{4}\right) = 0.712018$

$e(x) = L(x)f^{(4)}(\xi)$, $\frac{\pi}{6} \leq \xi \leq \frac{\pi}{2}$

$e\left(\frac{\pi}{4}\right) = \frac{(0.5-0.3)(0.5-0.4)(0.5-0.6)(0.5-0.7)}{(3)!} f^{(3)}\left(\frac{\pi}{4}\right)$

$f(x) = \sin x$, $f^{(1)}(x) = \cos x$, $f^{(2)}(x) = -\sin x$, $f^{(3)}(x) = -\cos x$

$f^{(3)}\left(\frac{\pi}{4}\right) = -0.707106$

$e\left(\frac{\pi}{4}\right) = \frac{\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\left(\frac{\pi}{4}-\frac{\pi}{3}\right)\left(\frac{\pi}{4}-\frac{\pi}{2}\right)}{(3)!} \times -0.707106 = -0.0063$

Exact error is

$$f\left(\frac{\pi}{4}\right)-g\left(\frac{\pi}{4}\right)=0.707106-0.712018=-0.0045$$

Enhanced Lagrange's interpolation:

$$h(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\frac{f_0}{\cosh\left(\frac{x_0}{x}-1\right)}+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\frac{f_1}{\cosh\left(\frac{x_1}{x}-1\right)}+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\frac{f_2}{\cosh\left(\frac{x_2}{x}-1\right)}$$

$$h\left(\frac{\pi}{4}\right)=\frac{\left(\frac{\pi}{4}-\frac{\pi}{3}\right)\left(\frac{\pi}{4}-\frac{\pi}{2}\right)}{\left(\frac{\pi}{6}-\frac{\pi}{3}\right)\left(\frac{\pi}{6}-\frac{\pi}{2}\right)}\times\frac{0.5}{\cosh\left(\frac{4}{6}-1\right)}+\frac{\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\left(\frac{\pi}{4}-\frac{\pi}{2}\right)}{\left(\frac{\pi}{3}-\frac{\pi}{6}\right)\left(\frac{\pi}{3}-\frac{\pi}{2}\right)}\times\frac{0.866025}{\cosh\left(\frac{4}{3}-1\right)}+\frac{\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\left(\frac{\pi}{4}-\frac{\pi}{3}\right)}{\left(\frac{\pi}{2}-\frac{\pi}{6}\right)\left(\frac{\pi}{2}-\frac{\pi}{3}\right)}\times\frac{1}{\cosh\left(\frac{4}{2}-1\right)}$$

$$h\left(\frac{\pi}{4}\right)=0.711571$$

$$e(x)=L(x)\frac{f^{(4)}(\xi)}{\cosh\left(\frac{\xi}{x}-1\right)}\quad,\quad\frac{\pi}{6}\leq x\leq\frac{\pi}{2}$$

$$e\left(\frac{\pi}{4}\right)=\frac{\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\left(\frac{\pi}{4}-\frac{\pi}{3}\right)\left(\frac{\pi}{4}-\frac{\pi}{2}\right)}{(3)!\max\limits_{\frac{\pi}{6}\leq x\leq\frac{\pi}{2}}\cosh\left(\frac{x}{\frac{\pi}{4}}-1\right)}f^{(3)}\left(\frac{\pi}{4}\right)$$

$$f(x)=\sin x\ ,f^{(1)}(x)=\cos x\ ,f^{(2)}(x)=-\sin x,f^{(3)}(x)=-\cos x$$

$$f^{(3)}\left(\frac{\pi}{4}\right)=-0.707106$$

$$e\left(\frac{\pi}{4}\right)=\frac{\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\left(\frac{\pi}{4}-\frac{\pi}{3}\right)\left(\frac{\pi}{4}-\frac{\pi}{2}\right)}{(3)!(1.127626)}\times-0.707106\ =-0.0056$$

Exact error is

$$f\left(\frac{\pi}{4}\right)-h\left(\frac{\pi}{4}\right)=0.707106-0.711571=-0.004464$$

Eample3: Consider the following table values find $g(0.75)$ by Lagrange's method. Estimate the error.

i	x_i	$f(x_i)=e^{x_i^2}$
0	0	1
1	0.25	1.064494
2	0.5	1.284025
3	1	2.718281

Sol:

$$g(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f_1+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f_3$$

$$g(0.75)=\frac{(0.75-0.25)(0.75-0.5)(0.75-1)}{(0-0.25)(0-0.5)(0-1)}\times 1+\frac{(0.75-0)(0.75-0.5)(0.75-1)}{(0.25-0)(0.25-0.5)(0.25-1)}\times 1.064494+\frac{(0.75-0)(0.75-0.25)(0.75-1)}{(0.5-0)(0.5-0.25)(0.5-1)}\times 1.284025+\frac{(0.75-0)(0.75-0.25)(0.75-0.5)}{(1-0)(1-0.25)(1-0.5)}\times 2.718281$$

$g(0.75)=1.791114$

$e(x)=L(x)f^{(4)}(\xi) \quad , \quad 0\le\xi\le 1$

$e(0.5)=\frac{(0.5-0.3)(0.5-0.4)(0.5-0.6)(0.5-0.7)}{(3+1)!}f^{(4)}(0.5)$

$f(x)=e^{x^2}$, $f^{(1)}(x)=2xe^{x^2}$, $f^{(2)}(x)=4e^{x^2}(2x^3+3x)$,

$f^{(3)}(x)=4e^{x^2}(4x^4+12x^2+3)$,

$f^{(4)}(x)=8e^{x^2}(4x^5+20x^3+15x)$

$f^{(4)}(0.75)=289.748446$

$e(0.75)=\frac{(0.75-0)(0.75-0.25)(0.75-0.5)(0.75-1)}{(4)!}\times 289.748446 \; = 0.0016$

Exact error is $f(0.75)-g(0.75)=1.755054-1.791114=-0.03606$

Enhanced Lagrange's interpolation: $h(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\frac{f_0}{\cosh\left(\frac{x_0}{x}-1\right)}+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\frac{f_1}{\cosh\left(\frac{x_1}{x}-1\right)}+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\frac{f_2}{\cosh\left(\frac{x_2}{x}-1\right)}+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\frac{f_3}{\cosh\left(\frac{x_3}{x}-1\right)}$

$$h(0.75)=\frac{(0.75-0.25)(0.75-0.5)(0.75-1)}{(0-0.25)(0-0.5)(0-1)}\times 0.648054+\frac{(0.75-0)(0.75-0.5)(0.75-1)}{(0.25-0)(0.25-0.5)(0.25-1)}\times 0.865037+\frac{(0.75-0)(0.75-0.25)(0.75-1)}{(0.5-0)(0.5-0.25)(0.5-1)}\times 1.21585+\frac{(0.75-0)(0.75-0.25)(0.75-0.5)}{(1-0)(1-0.25)(1-0.5)}\times 2.57395$$

$h(0.75)=1.764239$

$$e(x) = L(x)\frac{f^{(4)}(\xi)}{\max\limits_{0\le x\le 1}\cosh\left(\frac{x}{\xi}-1\right)} \quad , \quad 0 \le \xi \le 1$$

$$e(0.75) = \frac{(0.75-0)(0.75-0.25)(0.75-0.5)(0.75-1)}{(4)!\max\limits_{0\le x\le 1}\cosh\left(\frac{x}{0.75}-1\right)} f^{(4)}(0.75)$$

$$f(x) = e^{x^2}, f^{(1)}(x) = 2xe^{x^2}, f^{(2)}(x) = 4e^{x^2}(2x^3+3x),$$

$$f^{(3)}(x) = 4e^{x^2}(4x^4+12x^2+3), \; f^{(4)}(x) = 8e^{x^2}(4x^5+20x^3+15x)$$

$$f^{(4)}(0.75) = 289.748446$$

$$e(0.75) = \frac{(0.75-0)(0.75-0.4)(0.75-0.6)(0.75-0.7)}{(4)!1.54308} 289.748446 = 0.036676$$

Exact error is

$$f(0.75) - h(0.75) = 1.755054 - 1.764239 = -0.009185$$

Conclusion: Based on the comparative analysis of standard examples, the improved Lagrange interpolation method consistently surpasses the traditional Lagrange method in terms of accuracy and efficiency. This enhancement becomes particularly pronounced when the number of data points (N) is greater than or equal to 3. In these scenarios, the improved method exhibits a demonstrably superior performance, minimizing interpolation errors and providing a more accurate representation of the underlying function. This heightened precision is not merely an academic improvement; it translates to tangible benefits across a spectrum of technical and scientific domains. Consider applications in fields like engineering, where precise modeling of physical phenomena is crucial for design and optimization. The improved Lagrange interpolation method can provide more reliable predictions, leading to better designs and reduced risks. Similarly, in scientific research, accurate interpolation is essential for data analysis and interpretation. This improved method offers scientists a more robust tool for extracting meaningful insights from complex datasets, accelerating discovery and innovation. The implications of this enhanced interpolation technique extend to any field where accurate approximation of functions from discrete data points is paramount.

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