Gamma-medial BKU-algebras

***S. M.Mostafa**[1]* , Amany M. menshawy[2] , Ola Wageeh Abd El – Baseer[3]

1,2,3Department of Mathematics, Faculty of Education, Ain Shams University, Roxy, Cairo, Egypt.

Abstract. In this paper, we introduce new algebraic structures, called a Γ-BKU-algebras, which is a generalization of a KU- algebras and discuss the basic properties of Γ-BKU-algebras, Moreover, the notion of medial Γ-BKU -algebras is introduced. Several theorems are stated and proved. The image and pre-image of medial -Γ-ideals are defined and how the homomorphic images and pre-images of medial -Γ ideals become medial -Γ-ideals in Γ-BKU-algebras is studied as well. Finally, we construct some algorithms applied to Γ-medial-ideal in Γ-BKU-algebras .

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1. Introduction

The notion of BCK-algebras was proposed by Iseki [2]in 1966. In [3], Iseki introduced the notion of a BCI-algebra which is a generalization of BCK-algebra . Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK / BCI-algebras and their relationship with other structures including lattices and Boolean algebras. There is a great deal of literature which has been produced on the theory of BCK/BCI-algebras, in particular, emphasis seems to have been put on the ideal theory of BCK/BCI-algebras. For the general development of BCK/BCI-algebras the ideal theory plays an important role. In [5] J.Meng and Y.B.Jun studied medial BCI-algebras. In [6] S.M.Mostafa, Y.B.Jun and EL-menshawy introduce the notion of medial ideals in BCI-algebras, they state the fuzzification of medial ideals and investigate its properties.

In [1,4,7,8 ,9 ,10] four classes of new algebraic structures which are called KU/ PKU/KK/PU –algebras are introduced. It is known that the class of KU-algebras is a proper subclass of the class of PKU/KK/PU-algebras.

In this paper ,we introduce new algebraic structures, called a Γ-BKU-algebra, which is a generalization of a KU- algebra and discuss the basic properties of Γ-BKU-algebras, Moreover, the notion of medial Γ-BKU -algebras is introduced. Several theorems are stated and proved. The image and pre-image of medial -Γ-ideals are defined and how the homomorphic images and pre-images of medial -Γ ideals become medial -Γ-ideals in Γ-BKU-algebras is studied as well. Finally, we construct some algorithms applied to Γ-medial-ideal in Γ-BKU-algebras.

2. Preliminaries

First, we recall certain definitions from [1,4,7,8 ,9 ,10] that are required in the paper

Definition 2.1 .Let (X; $*$, 0) be a set with a binary operation ($*$) and a constant (0). Then (X; $*$, 0) is called a KU-algebra if it satisfies the following axioms:

For all $x, y, z \in$ X,

(KU-1) $(x*y)*((y*z)*(x*z)) = 0$,

(KU-2) $x*0 = 0$,

(KU-3) $0*x = x$,

(KU-4) $x*y = 0$ *and* $y*x = 0$ *imply* $x = y$.

In a KU-algebra X, we define a binary relation ($\leq$) by putting $x \leq y$ if and only if $y*x = 0$.

Remark.2.2 Let X be a nonempty set with a binary relation ($\leq$) on X and a fixed element (0) of X. Then X is a KU-algebra if and only if satisfies that: for all $x, y, z \in$ X,

(KU-1') $(y*z)*(x*z) \leq (x*y)$,

(KU-2') $0 \leq x$,

(KU-3') $x \leq y$ *and* $y \leq x$ *implies* $x = y$,

(KU-4') $x \leq y$ *if and only if* $y*x = 0$ *imply* $x = y$.

Form now on, for any KU-algebra (X; $*$, 0) , ($*$) and ($\leq$) are called a KU-operation and KU-ordering on X respectively .

Example2.3 .Let X = {0, 1, 2, 3, 4} in which ($*$) be defined by:

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	3	4
2	0	1	0	3	4
3	0	0	0	0	4
4	0	0	0	0	0

Then (X; $*$, 0) is a KU-algebra.

Proposition 2.4.Let (X; $*$, 0) be a KU-algebra, the following hold:

For all $x, y, z \in$ X,

1. $z*(x*z)=0$,
2. $x \le y$ *implies* $y*z \le x*z$ *and* $z*y \le z*x$,
3. $z*(y*x)=y*(z*x)$,
4. $y*((y*x)*x)=0$.

Definition 2.5.Let (X; $*$, 0) be a KU-algebra and I be a nonempty subset of X. I is called a KU-ideal of X if it satisfies:

(IKU_1) $0 \in I$,

(IKU_2) $x*(y*z) \in I$ *and* $y \in I$ *imply* $x*z \in I$, for all $x, y, z \in$ X.

Definition2.6. A ***PU***-algebra is a non-empty set X with a constant 0 $\in$ X and a binary operation $*$ satisfying the following conditions:

(KU-2) $x*0=0$,

(PU-1) (x $*$ z) $*$ (y $*$ z) = y $*$ x for any x, y, z $\in$ X.

On X we can define a binary relation " $\le$ " by: x $\le$ y if and only if y $*$ x = 0.

Example 2.7.Let X = {0, 1, 2, 3, 4} in which $*$ is defined by

$*$	0	1	2	3	4
0	0	1	2	3	4
1	4	0	1	2	3
2	3	4	0	1	2
3	2	3	4	0	1
4	1	2	3	4	0

Then (X, $*$, 0) is a ***PU*-algebra.**

Proposition 2.8.In a ***PU***-algebra (X, $*$, 0) the following hold for all x, y, z $\in$ X:

(a) x $*$ x = 0.

(b) (x $*$ z) $*$ z = x.

(c) x $*$ (y $*$ z) = y $*$ (x $*$ z).

(d) x $*$ (y $*$ x) = y $*$ 0.

(e) (x $*$ y) $*$ 0 = y $*$ x.

(f) If $x \leq y$, then $x * 0 = y * 0$.

(g) $(x * y) * 0 = (x * z) * (y * z)$.

(h) $x * y \leq z$ if and only if $z * y \leq x$.

(i) $x \leq y$ if and only if $y * z \leq x * z$.

(j) In a ***PU***-algebra $(X, *, 0)$, the following are equivalent:

(1) $x = y$, (2) $x * z = y * z$, (3) $z * x = z * y$.

(k) The right and the left cancellation laws hold in X.

(l) $(z * x) * (z * y) = x * y$,

(m) $(x * y) * z = (z * y) * x$.

(n) $(x * y) * (z * u) = (x * z) * (y * u)$ for all x, y, z and $u \in X$.

Definition2.10 .An algebra $(X, \cdot, 0)$ is called Pseudo KU-algebras (PKU-algebra) or (KK-algebra), if satisfying identities:

(KU-1) $(x * y) * ((y * z) * (x * z)) = 0$,

(KU-3) $0 * x = x$,

(KU-4) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$.

Example2.11.Let X = {0, 1, 2, 3} be a set with binary operations $\cdot$ and *
defined as follows:

•	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	0	1	0	2
3	0	1	0	0

*	0	1	2	3
0	0	1	2	3
1	0	0	3	0
2	0	0	0	0
3	0	3	3	0

One can see that (X, •, 0) is a PKU-algebra but not a KU-algebra, but $(X, *, 0)$ is both a KU-algebra and a PKU-algebra.

For elements x and y in a PKU-algebra X, $x*y = 0$ if and only if $y \leq x$. Then $(X, \leq)$ is a partial ordered set.

3. Γ -BKU-algebras.

In this section is to introduce the notion of Γ-BKU-algebras.

Definition 3.1. Let X and Γ be any nonempty sets. The structure $(\Gamma, X, 0)$ is called a Γ- BKU –algebra. If there exists a mapping $X \times \Gamma \times X \to X$ written as (x, γ, y) by $y\gamma x$, that satisfies the following condition axioms:

(γku_1) $(y\alpha z)\beta[(z\alpha x))\gamma(y\alpha x)] = 0$,

(γku_2) $0\alpha x = x$,

(γku_3) $x\alpha y = 0$ and $y\alpha x = 0$ implies $x = y$, for all $x, y, z \in X$, $\gamma, \beta, \alpha \in \Gamma$

On a Γ-BKU-algebra $(X, \Gamma, 0)$ we can define a binary relation $\leq$ on X by putting:

$x \leq y \Leftrightarrow y\gamma x = 0$.

Then $(\Gamma, X, 0)$ is a Γ- BKU –algebra if and only if it satisfies that :

$(\gamma ku_1')$ $[(z\alpha x)\beta(y\alpha x)] \leq y\alpha z$,

$(\gamma ku_2')$ $0 * x = x$,

$(\gamma ku_3')$ $x \leq y$ and $y \leq x$ implies $x = y$, for all $x, y, z \in X$, $\gamma, \beta, \alpha \in \Gamma$

Example 3.2 Let $M = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$ be nonempty set of binary operations defined below

α	0	a	b	c
0	0	a	b	c
a	0	0	a	c
b	0	0	0	c
c	0	a	b	c

β	0	a	b	c
0	0	a	b	c
a	0	0	a	a
b	0	a	0	b
c	0	a	0	0

γ	0	a	b	c
0	0	a	b	c
a	0	0	c	0
b	0	0	0	0
c	0	c	c	0

Clearly M is Γ - BKU –algebra.

Example3.3. Let $(\Gamma, X, 0)$ be an arbitrary BKU-algebra and Γ any nonempty set.
Define a mapping $X \times \Gamma \times X \to X$ by $y\gamma x \to y * x$ for all $x, y, z \in X$ and $\gamma \in \Gamma$.
It is easy to see that $(\Gamma, X, 0)$ is Γ - BKU -algebra. Indeed,

(γku_1) $(y\alpha z)\beta[(z\alpha x))\gamma(y\alpha x)] = 0$

(γku_2) $0\gamma x = x$,

(γku_3) $x\gamma y = 0$ and $y\gamma x = 0$ implies $x = y$,

Example 3.4.Let X = {0, a, b, c} in which (*) be defined by the following table:

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	c

and $\Gamma \neq \Phi$ Define a mapping $X \times \Gamma \times X \to X$ by $b\gamma a = a * b$ for all $a, b, c \in X$ and $\gamma \in \Gamma$. Then X is Γ - BKU –algebra, but X it not BKU –algebra since $0 * a = 0 \neq a$.

Lemma3.5: In Γ - BKU –algebra X , we have ,for all x, y, z $\in$ X, $\gamma, \beta, \alpha \in \Gamma$

$$z\alpha(y\beta x) = y\alpha(z\beta x) \text{ , for all } x, y, z \in X \text{ and } \gamma, \beta \in \Gamma$$

Proof : From (γku_1) we get $(0\alpha z)\beta[(z\alpha x))\gamma(0\alpha x)] = 0$

, this implies $z\beta[(z\alpha x)\gamma(x)] = 0$, i.e $[(z\alpha x)\gamma(x)] \leq z$ ------------------- (a)

Making use of (a) and ($\gamma ku_1'$) , we get $z\alpha(y\beta x) \leq [(z\alpha x)\alpha x]\beta[y\alpha x] \leq y\alpha(z\beta x)$

Since x, y, z $\in$ X, $\gamma, \beta, \alpha \in \Gamma$ are arbitrary , interchanging y and z in the above inequality , we obtain $y\alpha(z\beta x) \leq z\alpha(y\beta x)$, By (γku_4) , we get $z\alpha(y\beta x) = y\alpha(z\beta x)$.

Lemma 3.6.Let X Γ - BKU –algebra, then $y\alpha x = y\beta x$ for any $x, y \in X$ and $\alpha, \beta \in \Gamma$
Proof . Let $x, y \in X$ and $\alpha, \beta \in \Gamma$,then $y\alpha x = y\alpha(0\beta x) = 0\alpha(y\beta x) = y\beta x$

Proposition3.7 3.3. For any Γ - BKU -algebra X, we have the following properties:
For all x, y, z $\in$ X, $\gamma, \beta, \alpha \in \Gamma$

(1) $x\gamma x = 0$

(2) $y\alpha((y\beta x)\gamma x) = 0$

(3)x $\leq$ y implies $y\gamma z \leq x\gamma z$ and $z\gamma x \leq z\gamma y$

Proof. Let x, y, z $\in$ X and $\gamma, \beta, \alpha \in \Gamma$

(1) Put in (γBku_1) y=z=0 and using (γBku_2) ,we get
$0 = (0\gamma 0)\gamma[(0\gamma x))\gamma(0\gamma x)] = 0\gamma(x\gamma x) = x\gamma x$

(2) $y\alpha((y\beta x)\gamma x) = (y\beta x)\alpha(y\gamma x) = (y\gamma x)\alpha(y\gamma x) = 0$

(3) Since x $\leq$ y implies $y\gamma x = 0$, we obtain by (γBku_1) $(y\gamma x)\gamma[(x\gamma z))\gamma(y\gamma z)] = 0$, hence $0\gamma[(x\gamma z))\gamma(y\gamma z)] = 0$ implies $[(x\gamma z))\gamma(y\gamma z)] = 0$, i.e $(x\gamma z) \leq (y\gamma z)$

Proposition3.8 3.3. If X is Γ - BKU -algebra , then $(y\alpha x)\gamma 0 = (y\gamma 0)\alpha(x\gamma 0)$ for any $x, y \in X$ and $\alpha, \gamma \in \Gamma$

Proof. Let $x, y \in X$ and $\alpha, \beta, \delta, \gamma, \in \Gamma$, then

$$\begin{aligned}
&(y\alpha x)\gamma 0 = (y\alpha x)\gamma((y\beta 0)\delta(y\beta 0)) = \\
&(y\beta 0)\gamma((y\alpha x)\delta(y\beta 0)) = \\
&(y\beta 0)\gamma(((y\alpha x)\delta(x\gamma(y\beta x))) = \\
&(y\beta 0)\gamma(x\delta((y\alpha x)\gamma(y\beta x))) = \\
&(y\beta 0)\gamma(x\delta 0) = (y\gamma 0)\alpha(x\gamma 0)
\end{aligned}$$

4. Γ *-medial BKU-algebra*

Definition4.1. An algebra $(X,*,0)$ of type (2, 0) is called a Γ-medial BKU-algebra if it satisfying the following condition: $(x\gamma y)\alpha(z\ \beta\, u)=(x\gamma z)\alpha(y\ \beta\ u)$,for all x,y,z and $u\in X$ and $\alpha,\beta,\gamma,\in\Gamma$

Lemma.4.2 If X is Γ-medial BKU-algebra, then $(x\gamma y)=(y\gamma x)\alpha 0$, for any $x,y\in X$ and $\alpha,\gamma\in\Gamma$

Proof . Let $x,y\in X$ and $\alpha,\gamma\in\Gamma$ .Then

$$(x\gamma y)=0\alpha\,(x\gamma y)=(y\gamma y)\alpha\,(x\gamma y)=(y\gamma x)\alpha\,(y\gamma y)=(y\gamma x)\alpha 0$$

Example4.3 3.12. Let X := {0, 1, 2, 3} be a set with the following table

*	0	1	2	3
0	0	1	2	3
1	2	0	3	1
2	1	3	0	2
3	3	2	1	0

Define $\Gamma\neq\Phi$ and a mapping $X\times\Gamma\times X\to X$ by $y\gamma x\to y*x$ for all $x,y,z\in X$ and $\gamma\in\Gamma$. It is easy to see that $(\Gamma,X,0)$ is Γ- medial BKU -algebra.

Example4.4 3.13. Let X := {0, 1, 2, 3,4,5} be a set with the following table

*	0	1	2	3	4	5
0	0	1	2	3	4	5
1	2	0	1	4	5	3
2	1	2	0	5	3	4
3	3	4	5	0	1	2
4	4	5	3	2	0	1
5	5	3	4	1	2	0

Define $\Gamma\neq\Phi$ and a mapping $X\times\Gamma\times X\to X$ by $y\alpha x\to y*x$ for all $x,y,z\in X$ and $\alpha,\beta,\gamma,\in\Gamma$ It is easy to see that $(\Gamma,X,0)$ is Γ- BKU –algebra, but not medial Γ- KU –algebra, since $(2\alpha 4)\beta(3\alpha 5)=3\alpha 5=5\neq(2\alpha 3)\beta(4\alpha 5)=5\alpha 1=3$

Lemma.4.5 If X is Γ-medial -BKU-algebra . Then $(x\alpha y)\gamma\, z=(z\alpha y)\gamma x$ for any $x,y\in X$ and $\alpha,\beta,\gamma,\in\Gamma$

Proof . Let $x,y\in X$ and $\alpha,\beta,\gamma,\in\Gamma$. Then

$$(x\alpha y)\gamma\, z=(z\gamma(x\alpha y))\beta 0=(x\gamma(z\alpha y))\beta 0=(z\alpha y)\gamma x\,.$$

Lemma.4.6 If X is Γ-medial BKU-algebra .Then $(y\alpha x)\gamma\, x=y$ for any $x,y\in X$ and $\alpha,\beta,\gamma,\in\Gamma$

Proof . Let $x,y\in X$ and $\alpha,\beta,\gamma,\in\Gamma$. Then

$(y\alpha x)\gamma\, x = (x\gamma(y\alpha x))\beta 0 = (y\gamma(x\alpha x))\beta 0 = (y\gamma 0)\beta 0 = (0\gamma y) = y$

Lemma4.7 If X is Γ-medial BKU-algebra .Then $(y\alpha 0)\gamma\, 0 = y$ for any $x, y \in X$ and $\alpha, \beta, \gamma, \in \Gamma$

Proof . Clear.

Lemma 4.8. If X is Γ -BKU-algebra . X is associative if and only if $x\alpha 0 = x$ for any $x \in X$ and $\alpha, \beta, \gamma, \in \Gamma$

Proof .If X is associative, and then $(x\alpha x)\beta x = x\alpha(x\beta x)$ .Which gives $0\beta x = x = x\alpha 0$ for any $x \in X$.

Conversely, assume $x\alpha 0 = x$ for any $x \in X$ and $\alpha, \beta \in \Gamma$. Then

$(z\alpha y)\beta x = (z\alpha y)\beta(x\alpha 0) = x\beta((z\alpha y)\alpha 0) = x\beta(z\alpha y) = z\beta(x\alpha y) =$
$z\beta(x\alpha(y\alpha 0)) = z\beta(y\alpha(x\alpha 0)) = z\beta(y\alpha x)$

Thus X is associative.

Lemma 4.9.Every medial Γ-BKU-algebra X satisfies the following property:

$(y\alpha x)\beta 0 = (y\alpha 0)\beta(x\alpha 0)$ for any $x, y \in X$ and $\alpha, \beta, \gamma, \delta \in \Gamma$

Proof. For any $x, y \in X$, we have

$$(y\alpha x)\beta 0 = (y\alpha x)\beta[(y\alpha 0)\delta(y\alpha 0)]$$
$$= (y\alpha 0)\beta[(y\alpha x)\delta(y\alpha 0)]$$
$$= (y\alpha 0)\beta[(y\alpha y)\delta(x\alpha 0))$$
$$= (y\alpha 0)\beta[0\delta(x\alpha 0)] = (y\alpha 0)\beta(x\alpha 0)$$

Corollary 4.10.Every associative Γ - BKU-algebra is medial.

Proof. By Lemma 4.8, $x\alpha 0 = x$ for any $x \in X$. For any $x, y \in X$, we have
$y\alpha x = y\alpha(x\beta 0) = x\alpha(y\beta 0) = (x\alpha y)\beta 0 = (x\alpha y)$.

It follows from Lemma 3.16 that X is a medial Γ-BKU-algebra

Lemma4.11.A Γ-KU-algebra X is medial if and only if it satisfies one of the following conditions: for any $x, y, z \in X$ and $\alpha, \beta, \gamma, \delta \in \Gamma$

$x, y \in X$ and $\alpha, \beta, \gamma, \delta \in \Gamma$

(i) $x\alpha y = (y\alpha x)\beta 0$

(ii) $(z\alpha y)\beta x = (x\alpha y)\beta z$

(iii) $(y\alpha x)\alpha x = y$

(iv) $(y\alpha 0)\alpha 0 = y$

Proof .If Γ-BKU -algebra X is medial, then

$(y\alpha x)\beta 0 = (y\alpha x)\beta(y\alpha y)$
$= (y\alpha y)\beta(x\alpha y) = 0\beta(x\alpha y) = (x\alpha y)$

Let us assume (i) holds in X ,then

$(z\alpha y)\beta x = (x\beta(z\alpha y))\delta 0 = (x\beta(z\alpha y))\delta(x\beta x) = (z\beta(x\alpha y))\delta(z\beta z) = (x\alpha y)\beta z$.

Which proves (ii) The condition (ii) implies mediality . Indeed, we have ,

$(x\alpha y)\beta(z\alpha u) = ((z\alpha u)\alpha y)\beta x = ((y\alpha u)\alpha z)\beta x = (x\alpha z)\beta(y\alpha u)$

i.e $(x\alpha y)\beta(z\alpha u) = (x\alpha z)\beta(y\alpha u)$

Assume (i) hold, then

$(y\alpha x)\alpha x = ((x\alpha(y\alpha x))\alpha 0 = (y\alpha(x\alpha x))\alpha 0 = (y\alpha 0)\alpha 0 = 0\alpha y$ =y

Hence $(y\alpha x)\alpha x = y$, proving (iii). If we put $x := 0$ in in (iii) ,then

$(y\alpha 0)\alpha 0 = y$, which proves (iv). Suppose (iv) holds. Then by Lemma 3.1

$y\alpha x = ((y\alpha x)\beta 0)\delta 0 = ((y\alpha 0)\beta(x\alpha 0)))\delta 0 = (x\beta((y\alpha 0)\beta 0))\delta 0 = (x\alpha y)\beta 0$

Hence $y\alpha x = (x\alpha y)\beta 0$,which completes the proof.

Definition 4.12.A non empty subset M of a medial Γ-BKU-algebra X is said to be a Γ-ideal of X if it satisfies:

(M_1) $0 \in M$,

(M_2) $y\alpha z \in M$ and $y \in M$ imply $z \in M$ for all $y, z \in X$ and $\alpha \in \Gamma$

Definition 4.12.A non empty subset M of a medial Γ-BKU-algebra X is said to be a Γ-medial ideal of X if it satisfies:

(M_1) $0 \in M$,

(M_2) $(x\alpha y)\beta z \in M$ and $z\alpha y \in M$ imply $x \in M$ for all $x, y, z \in X$ and $\alpha, \beta \in \Gamma$

Example 4.13.Let X = {0, 1, 2, 3, 4, 5} be a set with a binary operation $*$ defined by the following table:

*	0	1	2	3	4	5
0	0	1	2	3	4	5
1	0	0	2	2	4	4
2	0	1	0	1	4	5
3	0	0	0	0	4	4
4	4	4	4	4	0	1
5	4	4	4	4	0	0

Define $\Gamma \neq \Phi$ and a mapping $X \times \Gamma \times X \to X$ by $y\alpha x \to y * x$ for all $x, y, z \in X$ and $\alpha, \beta, \gamma, \in \Gamma$ It is easy to see that $(\Gamma, X, 0)$ is Γ- BKU –algebra, we can prove that $(X, \Gamma, 0)$ is a Γ-BKU-algebra and A = {0, 1, 2, 3} is a Γ-medial-ideal of X.

Proposition4.14.Any ideal of a medial Γ-BKU -algebra is a Γ-medial ideal.
Proof. Let M be a ideal in a medial Γ-BKU -algebra X, such that $(x\alpha y)\beta z \in M$ and $z\beta y \in M$, for all $x, y, z \in X$, by (Lemma4.11.(ii)),we have $(z\alpha y)\beta x \in M$, $z\alpha y \in M$. But M is ideal; therefore $x \in M$ .Then M is a Γ- medial ideal.

Proposition 3.22.Any Γ- medial ideal of a Γ-BKU-algebra X must be a Γ- ideal but the converse is not true.

Proof .Let M be a Γ- medial ideal of a Γ-BKU-algebra X, such that $(x\alpha y)\beta z \in M$ and $z\alpha y \in M$, for all $x, y, z \in X$ $\alpha, \beta \in \Gamma$, (Lemma4.11.(ii)),we have $(z\alpha y)\beta x \in M$, $z\alpha y \in M$ .Then M is Γ-ideal. The last part is shown by the following example

Example3.23. Let X = {0, 1, 2, 3, 4} be a set with a binary operation $*$ defined by the following table:

*	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	1	4
2	0	1	0	3	4
3	0	0	2	0	4
4	0	1	0	3	0

Define $\Gamma \neq \Phi$ and a mapping $X \times \Gamma \times X \to X$ by $y\alpha x \to y * x$ for all $x, y, z \in X$ and $\alpha, \beta \in \Gamma$ It is easy to see that $(\Gamma, X, 0)$ is Γ- BKU –algebra, and $I = \{0,1,3\}$ is Γ- ideal, but not Γ-medial, since $(4\alpha 1)\beta 3 \in I$ *and* $3\alpha 1 \in I$, *but* $4 \notin I$.

5. Homomorphism of Γ-BKU-algebras

Definition5.1.Let (X , Γ ,0) and (Y , Γ ,0\`) be BKU-algebras , the mapping $f$ : (X , $\Gamma$ ,0) $\to$ (Y , $\Gamma$ ,0\`) is called a homomorphism if it satisfying f(x α y) = f (x) β f (y) for all x , y $\in$ X. The set {x$\in$X | f(x) = 0'} is called the Kernel of f denoted by *Kerf* .

Theorem 5.2.Let f : (X , Γ ,0) $\to$ (Y , Γ ,0\`) be a homomorphism of a Γ-BKU-algebra X into a Γ-BKU-algebra Y, then :
(1) f(0) = 0' .
(2) f is injective if and only if Ker f = {0}.

(3) $x \le y$ implies $f(x) \le f(y)$.

Proof: Assume that $f:(X,\Gamma,0) \to (Y,\Gamma,0`)$ is a Γ-BKU-homomorphism.

(1) Since $0*0 = 0$, then $f(0) = f(0\alpha 0) = f(0)\ \beta f(0) = 0'$.

(2) Suppose that f is injective and $x \in Kerf$. It follows that f (x) = 0'. Since $f(0) = 0$, so $f(x) = f(0)$. By assumption, x = 0. Thus $Kerf = \{0\}$.

Conversely, suppose that Ker $f = \{0\}$. Let x, y $\in$ X be such that $f(x) = f(y)$. Then we get that $f(x\alpha y) = f(x)\beta\ f(y) = 0'$ and $f(y\alpha x) = f(y)\ \beta f(x) = 0'$, thus $x\alpha y$, $y\alpha x \in Kerf$, this means that $x\alpha y = 0 = y\alpha x$. From ($Bku_3$), x = y, and shows that f is injective.

(3) Let $x \le y$. It follows that $y\ \alpha\ x = 0$. So, from (Theorem 5.2 (1)) implies $f(y)\ \beta f(x) = f(y\alpha\ x) = f(0) = 0'$. Hence $f(x) \le f(y)$.

Theorem 5.3.Let $f:(X,\Gamma,0) \to (Y,\Gamma,0`)$ be a homomorphism of a Γ-BKU-algebra X into a Γ-BKU-algebra Y, then :

(1) If I is an Γ-ideal of X, then f (I) is an Γ-ideal in Y.

(2) If J is a Γ- ideal in Y, then f^{-1} (J) is a Γ-ideal in X.

(3) Ker f is Γ-ideal of X.

Proof: Straightforward.

Theorem 3.4.Let $f:(X,\Gamma,0) \to (Y,\Gamma,0`)$be a homomorphism of a Γ-BKU-algebra X into a Γ BKU-algebra Y, then :

(1) If M is Γ- medial -ideal of X, then f (M) is an Γ-medial -ideal in Y .

(2) If J is a Γ- medial - ideal in Y, then f^{-1} (J) is a Γ medial -ideal in X .

(3) Ker f is a Γ- medial -ideal of X.

Proof: (1) Let M be Γ-medial -ideal of X. We see that $0 \in$ M, by theorem 5.2(1), $0' = f(0) \in f(M)$ so $0' \in f(M)$ Now, assume that $f((x\alpha y)\beta z) = f(x\alpha y)\beta f(z) = (f(x)\alpha f(y))\beta f(z) \in f(M)$ and $f(z*y) = (f(z)\alpha f(y)) \in f(M)$ it follows that $(x\alpha y)\beta z \in M$ and $z\alpha y \in M$, since M is an Γ-medial-ideal of X, it follows that $x \in M$ imply that $f(x) \in f(M)$

Hence $f(M)$ is Γ-medial-ideal of Y.

(4) Let J be an Γ-medial-ideal of Y. Then $0' \in J$, and hence $0 = f^{-1}(0') \in f^{-1}(J)$. Now, for any x, y , z $\in$ X, let $(x\alpha y)\beta z \in f^{-1}(J)$ and $(z\alpha y) \in f^{-1}(J)$

It follows that $f((x\alpha y)\beta z) = f(x\varepsilon y)\beta f(z) = (f(x)\alpha f(y))\beta f(z) \in J$ and $f(z\alpha y) = f(z)\alpha f(y) \in J$. Since J is an medial-ideal of Y, we obtain that $f(x) \in J$. Consequently $x \in f^{-1}(J)$, proving that $f^{-1}(J)$ is an Γ-medial-ideal of X.

(5) It is clear that Ker f $\subseteq$ X. Since f(0) = 0', so 0 $\in$ Ker f. For any x, y , z $\in$ X, let $(x\alpha y)\beta z \in Kerf$ and $z\alpha y \in Kerf$ .Then $f((x\alpha y)\beta z) = f(x\alpha y)\beta f(z) = (f(x)\alpha f(y))\beta f(z) = 0$ and $f(z\alpha y) = f(z)\alpha f(y) = 0$,which implies that $f(x) = 0$,i.e $x \in Kerf$ .Therefore $Kerf$ is a Γ-medial -ideal of X.

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Conclusion.

We have studied the medial Γ - BKU-algebras. Also we discussed few results of medial-Γ-ideal in Γ-BKU-algebras. The image and the pre- image of medial Γ-ideal in Γ - BKU-algebras under homomorphism are defined and how the image and the pre-image of medial Γ-ideal in Γ-BKU-algebras become medial Γ-ideal are studied. Moreover, the medial Γ-ideal is established. Furthermore, we construct some algorithms applied to medial Γ-ideal in Γ - BKU-algebras.

The main purpose of our future work is to investigate the fuzzy foldedness of medial Γ-ideal in Γ-BKU-algebras, cubic medial Γ-ideal, hyper Γ - BKU-algebras.

appendix

This appendix contains all necessary algorithms

A: Algorithm for Γ-BKU-algebras
Input (X set , Γ :set of binary operations)
Output (X is a Γ-BKU-algebra or not")
Begin
If $X = \Phi$ then go to (1.);
EndIf
If $0 \notin X$ then go to (1.);
EndIf
Stop: =false;
$i := 1$;
While $i \leq |X|$ and not (Stop) do
If $0 \alpha x_i \neq x_i \quad \forall \alpha \in \Gamma$ then
Stop: = true;
EndIf
$j := 1$
While $j \leq |X|$ and not (Stop) do
If $z\alpha(y\beta x) \neq y\alpha(z\beta x) \quad \forall \alpha, \beta \in \Gamma$ then
Stop: = true;
EndIf
EndIf
$k := 1$

While $k \le |X|$ and not (Stop) do
If $(y\alpha z)\beta[(z\alpha x))\gamma(y\alpha x)] \ne 0 \ \forall \alpha, \beta, \gamma \in \Gamma$ then
Stop: = true;
EndIf
EndIf While
EndIf While
EndIf While
If Stop then
Output ("X is not a Γ-BKU-algebra")
Else
Output ("X is a Γ-BKU-algebra")
EndIf

B: Algorithm for medial Γ-ideals
Input (X BKU-algebra, M subset of X);
Output ("M is an medial - Γ-ideals of X or not");
Begin
If $M = \Phi$ then go to (1.);
End If
If $0 \notin M$ then go to (1.);
End If
Stop: =false;
$i := 1$;
While $i \le |X|$ and not (Stop) do
$j := 1$
While $j \le |X|$ and not (Stop) do
$k := 1$
While $k \le |X|$ and not (Stop) do
If $(x_i \alpha y_j)\beta z_k \in M$ and $z_k \alpha y_j \in M$ then
If $x_i \notin M$ then
Stop: = true;
End If
End If
End While
End While
End While
If Stop then
Output ("M is is an medial Γ-ideals of X")
Else
(1.) Output ("M is not is an medial Γ-ideals of X")
End If
End

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